Stochastic Integration and Stochastic Differential Equations: a gentle introduction

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Intro: why Stochastic?

Brownian Motion/ Wiener process

Stochastic Integration

Stochastic DE's

Applications

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Deterministic ODE

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- $\omega(t)$ is random with mean 0
- $\omega(t)$ is *independent* of $\omega(s)$, $t \neq s$ $\Rightarrow$ "White noise"
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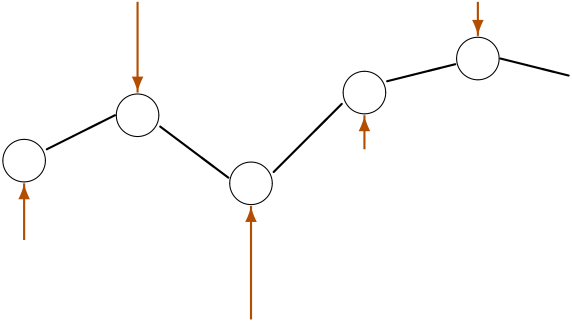
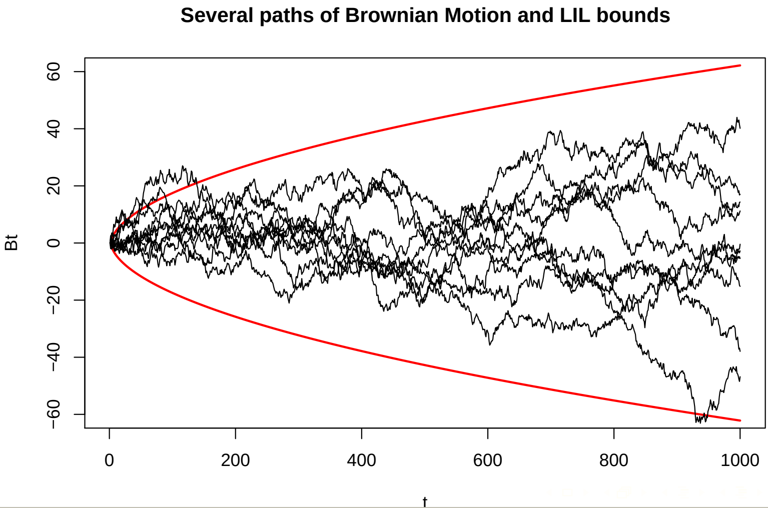
Strictly speaking, ω does not exist!

Brownian motion

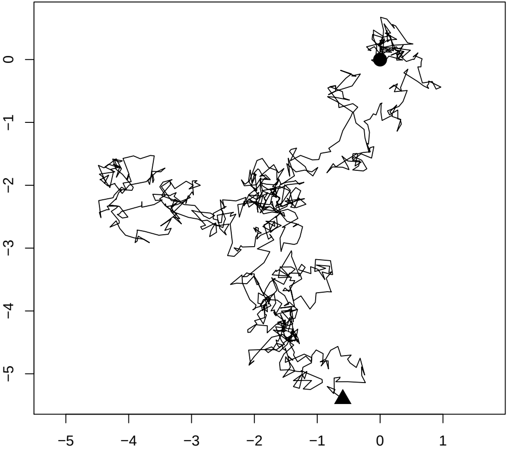


Intuitive: "Random walk", "Diffusion"
Stocks: you cannot predict the future behavior based on past performance

Brownian motion



2D Brownian motion

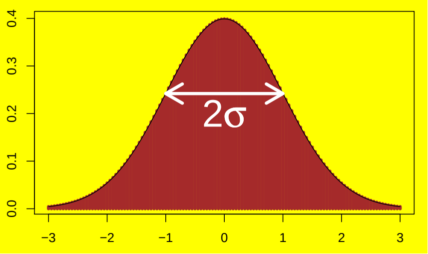


Brownian motion: Axioms

BM is continuous (but not smooth, see later!), $W(0) = 0$

Normality:

$$W(t+\Delta t)-W(t) \sim \mathit{Normal}(\texttt{mean} = 0, \texttt{variance}\ \sigma^2 = \Delta t)$$

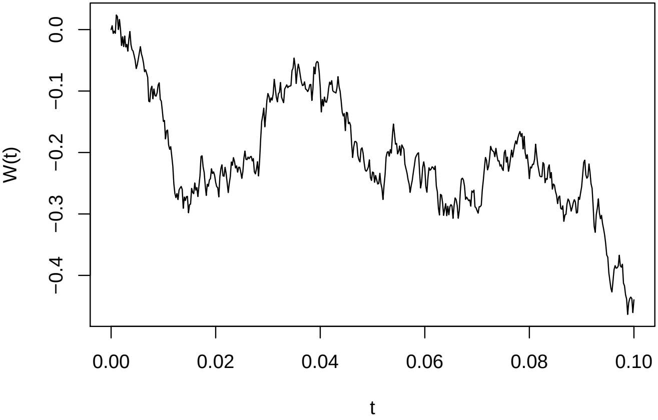


$$\sigma = \sqrt{\Delta t}$$

Normal because sum of many small, independent increments.

Brownian motion: Axioms

Independence:



Brownian motion: Axioms

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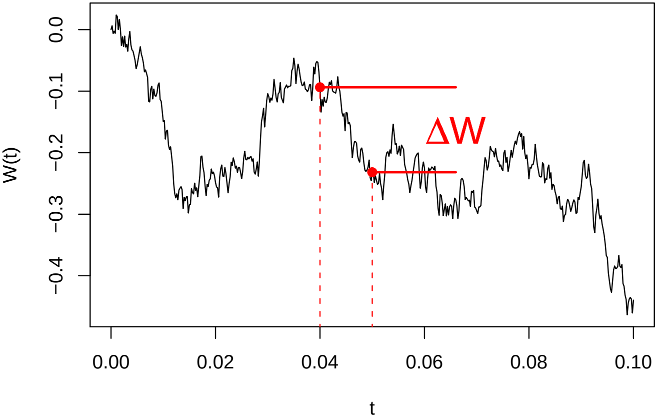

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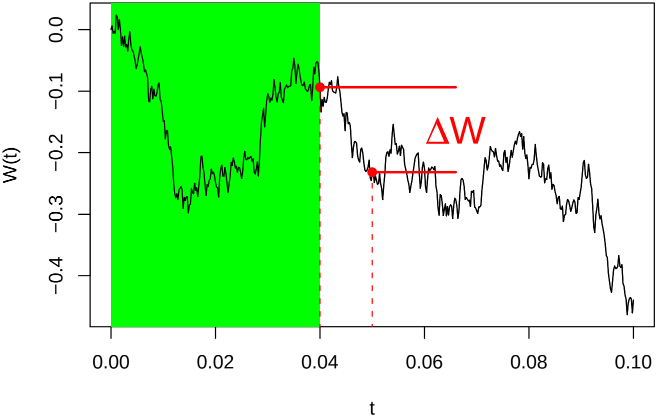
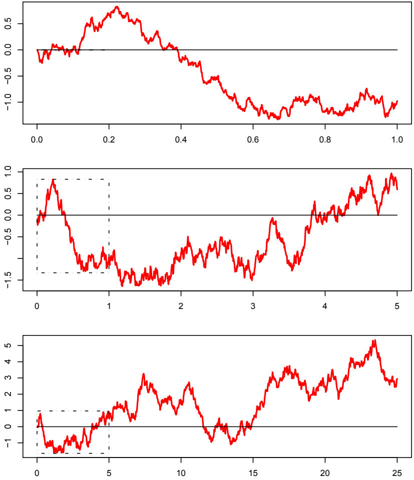


Fractal nature



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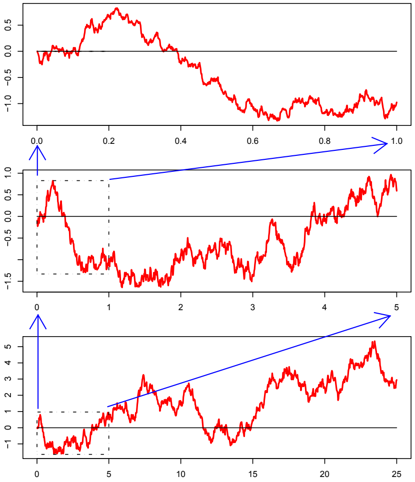

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- Riemann integral $$\int_0^T h(t)\,dt \approx \sum_{i=1}^n h(t_i^*)\Delta t_i$$
- Stieltjes integral $$\int_0^T h(t)\,dF(t) \approx \sum_{i=1}^n h(t_i^*)[F(t_{i+1}) - F(t_i)]$$

  where F has *finite variation*
  - when F' exists, then $\int_0^T h(t)\,dF(t) = \int_0^T h(t)F'(t)\,dt$

Stochastic Integration

- Try the same:

$$\int_0^T h(t)\, dW(t) \approx \sum_{i=1}^{n} h(t_i^*)[W(t_{i+1}) - W(t_i)]$$

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Itô

$$\sum_{i=1}^{n} h(t_i)[W(t_{i+1}) - W(t_i)]$$

Stratonovich

$$\sum_{i=1}^{n} h\left(\frac{t_i + t_{i+1}}{2}\right)[W(t_{i+1}) - W(t_i)]$$

Stochastic Integration: examples

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$$2\sum W(t_i)[W(t_{i+1}) - W(t_i)]$$

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- Hence, $\displaystyle\int_0^T W(t)\,dW(t) = \frac{W^2(T)}{2} - \frac{T}{2}$ (Itô)
- Wish it were easier?

Stochastic Differential Equations

$$X(t) = \int_0^t b(s, X(s))\, dW(s), \qquad t > 0$$

$$\Downarrow$$

$dX(s) = b(s, X(s))\, dW(s)$ - Stochastic differential

Full form

$$\begin{cases} dX(s) = a(s, X(s))\, ds + b(s, X(s))\, dW(s), \\ X(0) = X_0 \end{cases}$$

(can be random)

(note that dW/ds does not exist)

Itô Formula

Let $dX(t) = a(t, X(t))\, dt + b(t, X(t))\, dW(t)$

then, the chain rule is

$$dg(t, X(t)) = g_t dt + g_x dX(t) + g_{xx} \frac{b^2(t, X(t))}{2}\, dt$$ - "extra" term

heuristic: follows from Taylor series assuming $(dW(t))^2 = dt$

for example,

$$\begin{aligned} d(W^2(s)) &= W^2(s+ds) - W^2(s) \\ &= [W(s+ds) + W(s)][W(s+ds) - W(s)] \\ &= [W(s+ds) - W(s)][W(s+ds) - W(s)] \\ &\qquad +2W(s)[W(s+ds) - W(s)] \\ &= 2W(s)\,dW(s) + ds \end{aligned}$$

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Hence, $\displaystyle\int_0^t W(s)\,dW(s) = \frac{W^2(t)}{2} - \frac{t}{2}$

$dg(t, X(t)) = g_t dt + g_x dX(t) + g_{xx} \frac{b^2}{2} dt$

Exercise:

- $d(W^3(s)) = ?$

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- $d(W^3(s)) =?$
- $= 3W^2(s)\, dW(s) + 3W(s) ds$

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- $d(sW(s)) = ?$
- $= W(s) ds + s\, dW(s)$,

  therefore $\int_0^t s\, dW(s) = tW(t) - \int_0^t W(s)\, ds$

Solution

What does the solution look like?

Note that for Itô integrals, we always have

$$E \int_0^t h(s)\, dW(s) = 0 \qquad \text{Expected value}$$

$$E\left[\int_0^t h(s)\, dW(s)\right]^2 = \int_0^t E[h^2(s)]\, ds \qquad \text{Variance}$$

(can see using the Riemann sums).

For example, the result of $\int_0^t s\, dW(s)$ is a Normal random variable, with mean 0 and variance =

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For example, the result of $\int_0^t s\,dW(s)$ is a Normal random variable, with mean 0 and variance $= t^3/3$

Langevin equation

More realistic Brownian motion: resistance/friction

$$dX(t) = -\beta X(t)dt + \sigma\, dW(t)$$

$X(t) =$ particle velocity

Then, can obtain using Itô formula, integrating factor $e^{-\beta t}$

$$X(t) = e^{-\beta t} X_0 + \int_0^t e^{-\beta(t-s)}\, dW(s)$$

Stock price

$$dX(t) = rX(t)\,dt + \sigma X(t)\,dW(t)$$

can express $d\,log(X(t))$ and get

$$X(t) = exp\left[(r - \frac{\sigma^2}{2})t + \sigma W(t)\right]$$

Note that for both equations, expected value $E[X(t)]$ coincides with the solution of deterministic equation, here

$$d\overline{X}(t) = r\overline{X}(t)\,dt$$

A nonlinear equation

Consider a logistic equation for population dynamics: non-linear

$$dX(t) = rX(t)(1 - X(t))\,dt + \sigma X(t)\,dW(t)$$

Here, deterministic solution of

$$d\overline{X}(t) = r\overline{X}(t)(1 - \overline{X}(t))\,dt$$

is different from the mean of stochastic solutions

A nonlinear equation

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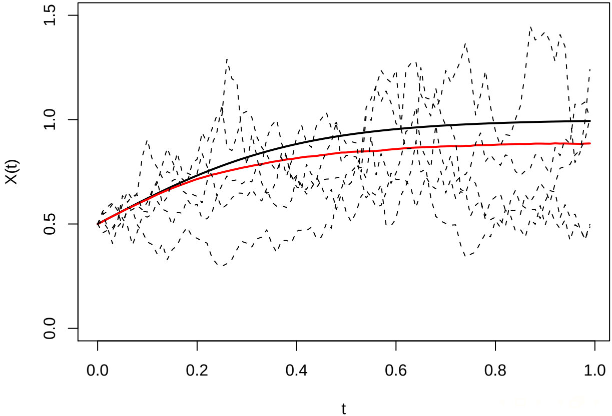


Math Finance

Hedge funds

Behind the veil

Big market losses provide insights into a volatile financial world

THEY are secretive, clever and often highly lucrative. But the veil of mystery that has shrouded hedge funds was partially lifted in the past fortnight after market turmoil left many of them with big losses—and anxious investors.

The Economist August 18th 2007

Among the hedge funds hardest hit were credit funds and those using a type of statistical arbitrage, known as long-short equity neutral. Stocks in these portfolios are picked assuming certain shares will rise and others will fall. In this case, the complex models that drive them were upended by the extreme market volatility. Four building-blocks of such models are stock valuations, quality, price momentum and earnings momentum. These usually offset each other, but when they all started suffering, the models went awry. Some of the world's biggest hedge funds all began selling the same things at the same time. "You had the proverbial camel trying to get through the eye of the needle," an analyst says.

Math Finance

Danger. Quant at work

Many funds will report to their investors in the coming weeks. Much depends on whether the pension funds, endowments and rich individuals investing in hedge funds hold their nerve. The "lock-up" periods for fund investors vary. Many allow redemptions only monthly or quarterly. August 15th was a big day for those who need to give 45 days notice before redeeming their stakes by the end of September.

The saga has damaged the image of computer-driven funds, generally so powerful that they can account for up to half of a stock exchange's daily trading volume. But there is no way the clocks will be turned back. "People aren't going to give up their computers and go back to insider information and tips," says David Harding, a fund manager in London.

It is also unclear who will gain from the turmoil—and someone always takes the

Math Finance

- Optimal control of investments
- Black-Scholes formula for option pricing
- Based on

$$dX(t) = rX(t)\,dt + \sigma X(t)\,dW(t)$$

and beyond

Other applications

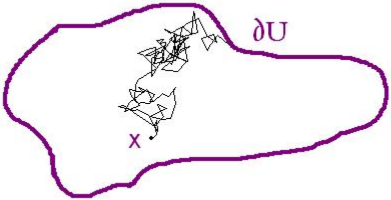
Boundary value problems: e.g. solving a Dirichlet problem

$\Delta f = 0$ (harmonic) in the region $U \in \mathbb{R}^n$

$f = g$ on the boundary ∂U, given g

a stochastic solution is

$$f(x) = E_x[g(\text{point of first exit from } U)],$$



E_x refers to Brownian motion started from x.

Other applications

Hydrology: porous media flow

$$\begin{cases} div(V) = 0 & \text{incompressible flow} \\ V = -K\nabla P & \text{Darcy's law} \end{cases}$$

V = velocity, P = pressure field
K = conductivity is *stochastic*

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Predator-prey models, Chemical reactions, Reservoir models ... ...

QUESTIONS?

THANK YOU!

www.nmt.edu/~olegm/SDE/ for references