Image deblurring as an inverse problem

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Can you read this?

- **Image:** F = a matrix of grayscale intensities (real numbers in a bounded interval), generally assumed unknown
- **Image blurring:** each pixel is spread out according to some matrix B, at the maximum of m pixels across and m pixels up/down.
- **Data:** a blurred image $G = \mathcal{K}_B F$ with the operator $\mathcal{K}_B$ based on the pixel blurring operation B applied to each pixel, and the results added up.
- **Deblurring problem:** Based on the given blurred image G (and, maybe, B), restore "as well as possible" the original image F

Matrix form: if F and G are turned into vectors f, g then $g = K_B f$ with some matrix $K = K_B$.

However, the matrix K is usually not invertible. Also, there's often noise involved:

$$g = Kf + e$$

Loss of information from blurring; some recovery is possible because the image F is redundant

I. Bayesian approach

Bayesian regularization

The image is treated as *Gauss-Markov Random Field* with the values at any site/pixel $s = (i, j)$ dependent on its neighbors.

Thus, the *prior* distribution for the image is

$$p(F) \;\propto \exp\left\{-\frac{1}{2\sigma^2}\sum_{\texttt{sites}\ s\sim t}(F(t)-F(s))^2\right\}$$

The *likelihood* of the data G given the image F

$$p(G|F) \;\propto \exp\left\{-\frac{1}{2\sigma_e^2}\sum_{\texttt{sites}\ s}\left[G(s)-\mathcal{K}F(\texttt{neighb. of}\ s)\right]^2\right\}$$

the parameters σ and σ_e control the degree of smoothness and overall sharpness of the posterior distribution

Posterior probability

Makes the Bayesian inversion, that is

$$\texttt{Posterior} \propto \texttt{Prior} \times \texttt{Likelihood}$$

$$\log p(F|G) = const - \frac{1}{2\sigma_e^2} \sum_{\texttt{sites } s \sim t} (F(t) - F(s))^2 -$$

$$-\frac{1}{2\sigma_e^2} \sum_{\texttt{sites } s} [G(s) - \mathcal{K}F(\texttt{neighb. of } s)]^2$$

Based on this (for example, finding its maximum, or maybe mean) we can produce an estimate of F

The posterior $p(F|G)$ is a multi-dimensional distribution that's hard to compute. Instead, we can *sample* its values (i.e. get the "typical" values of F).

Gibbs sampler:

To set up a Gibbs sampler, we start with random values for the parameters (in our case, values of F). Then, sample from the *full conditional posteriors*

$$P(\texttt{a parameter} | \texttt{ all other parameters, data } G, \sigma, \sigma_e)$$

In a simple single-site updating scheme, update F(s) based on the values of its neighbors.

Pressures to:

1) conform to its neighbors
2) conform to the data G

An evolution scheme which eventually migrates to "likely" F.

Metropolis algorithm

In order to draw from $P(F(s)|F(\texttt{neighbor sites}), G)$, we use **Metropolis algorithm**:

1. Draw a new candidate $f_{\texttt{new}}$ for $F(s)$
2. Compare the posteriors $post = p(F|G)$ with the old and the new F.
3. If the new value has higher posterior, then accept it.
   If lower, then accept it with probability $= post_{new}/post_{old}$

It's nice because the knowledge of proportionality constant in $p(F|G)$ is not required!

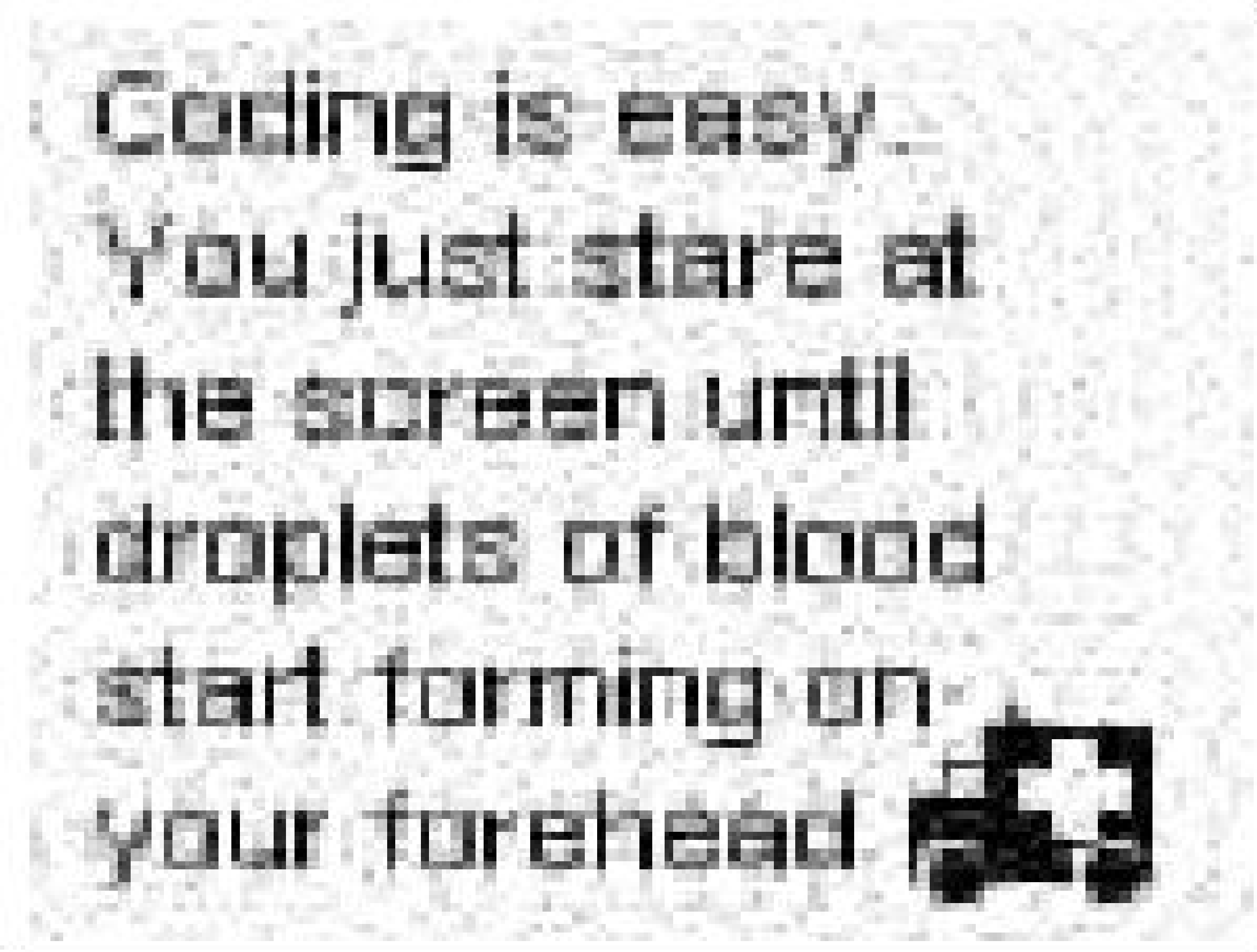
Furthermore, one can use *simulated annealing* based on Metropolis algorithm to find the "most likely" F.

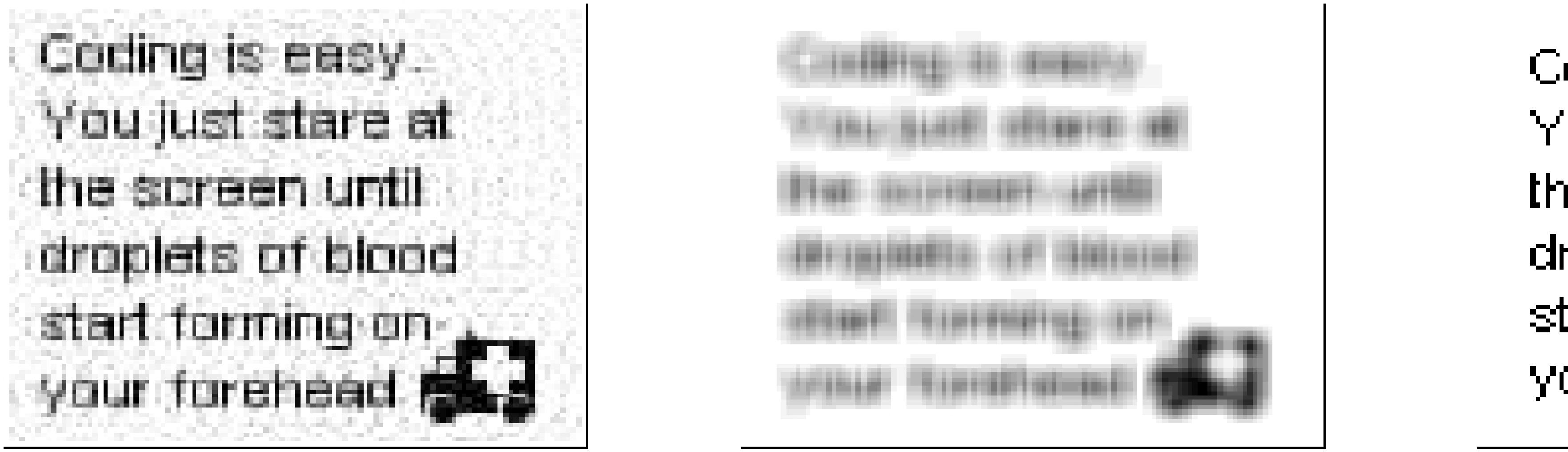
[Demo of Gibbs here]

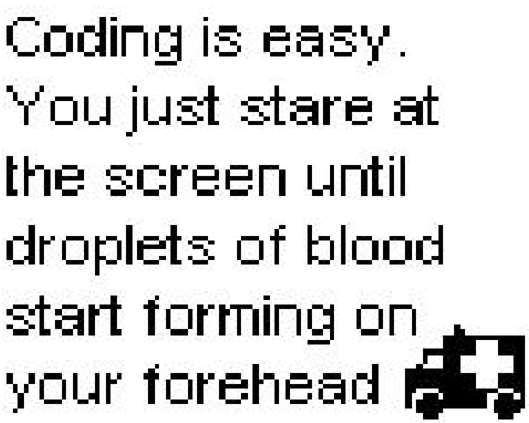
Some results: $\hat{F}, G, \mathcal{K}\hat{F}$



Some results:




Coding is easy.
You just stare at
the screen until
droplets of blood
start forming on
your forehead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II. Regularization approach

Regularization approach:

Instead of solving $\min_f ||Kf - g||$, let's solve

$$\min_f ||Kf - g||^2 + \alpha ||f||^2$$

i.e. penalize large values of f

Compute the *singular value decomposition (SVD)* of K:

$$K = UDV'$$

where U, V are orthogonal matrices, and $D = diag\{d_1, d_2, ..., d_N\}$ consists of *singular values* of K.

Let K^+ be a regularized inverse of K (see below) then, the deblurred image $\hat{F}$ is

$$\hat{F} = K^+ G = V D^+ U' G$$

where the inverse D^+ is taken using

$$d_i^+ = \frac{d_i}{\alpha^2 + d_i^2}$$

However, K is huge...

Note: if K is a *circulant* matrix,

$$K = \begin{bmatrix} c_0 & c_{n-1} & c_{n-2} & \cdots & c_2 & c_1 \\ c_1 & c_0 & c_{n-1} & \cdots & c_3 & c_2 \\ & & \ddots & & & \vdots \\ & & & \ddots & & \vdots \\ \cdots & & & & \ddots & \vdots \\ c_{n-1} & c_{n-2} & \cdots & & c_1 & c_0 \end{bmatrix}$$

then

$$K = WDW^*$$

where D is diagonal, and W is unitary, but now they have complex values. (W contains basis for Discrete Fourier Transform.)

Such K assumes the toroidal structure of the image (first row linked to the last row etc.)
D contains the Fourier coefficients for the first column of K. It is real-valued if the blur B is symmetric.

No worries, we can still do the regularization D^+:

$$F = K^+ G = W^* D^+ W G$$

As an added bonus, we use FFT (Fast Fourier Transform), that is multiplying by W is equivalent to making a FFT, and multiplying by W^* is equivalent to a ...

[Live demo here]

GEMAN AND YANG: NONLINEAR IMAGE RECOVERY WITH HALF-QUADRATIC REGULARIZATION

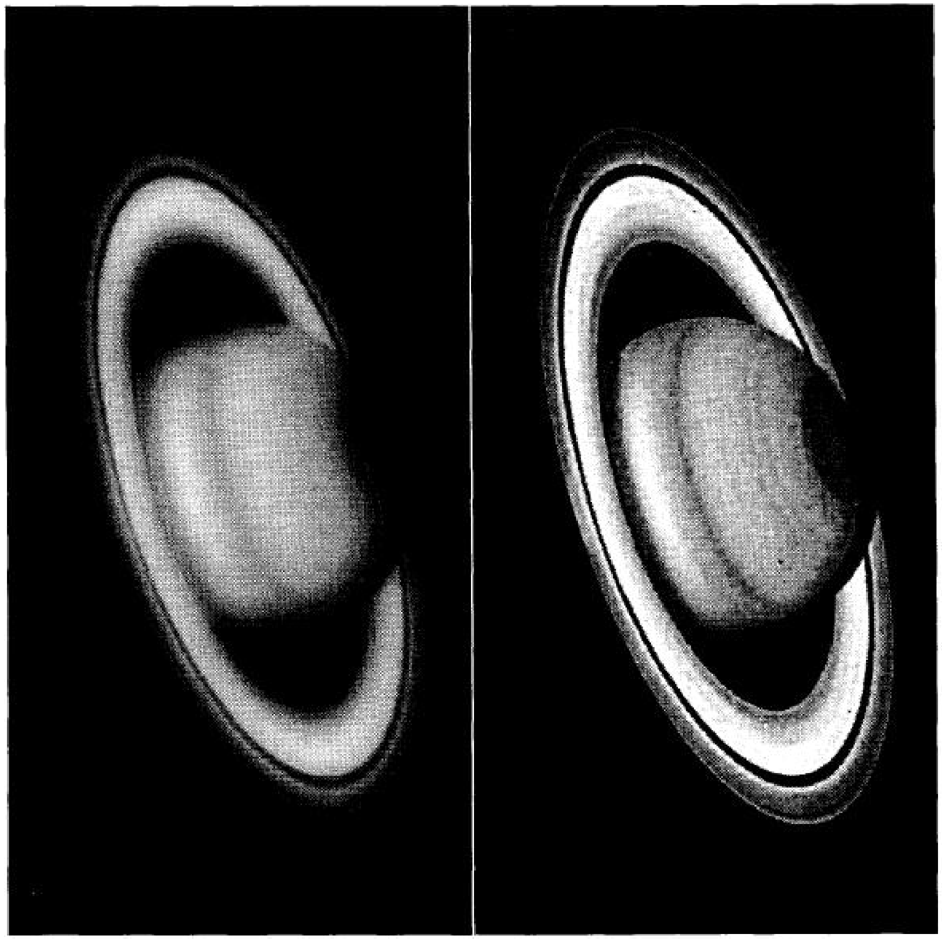
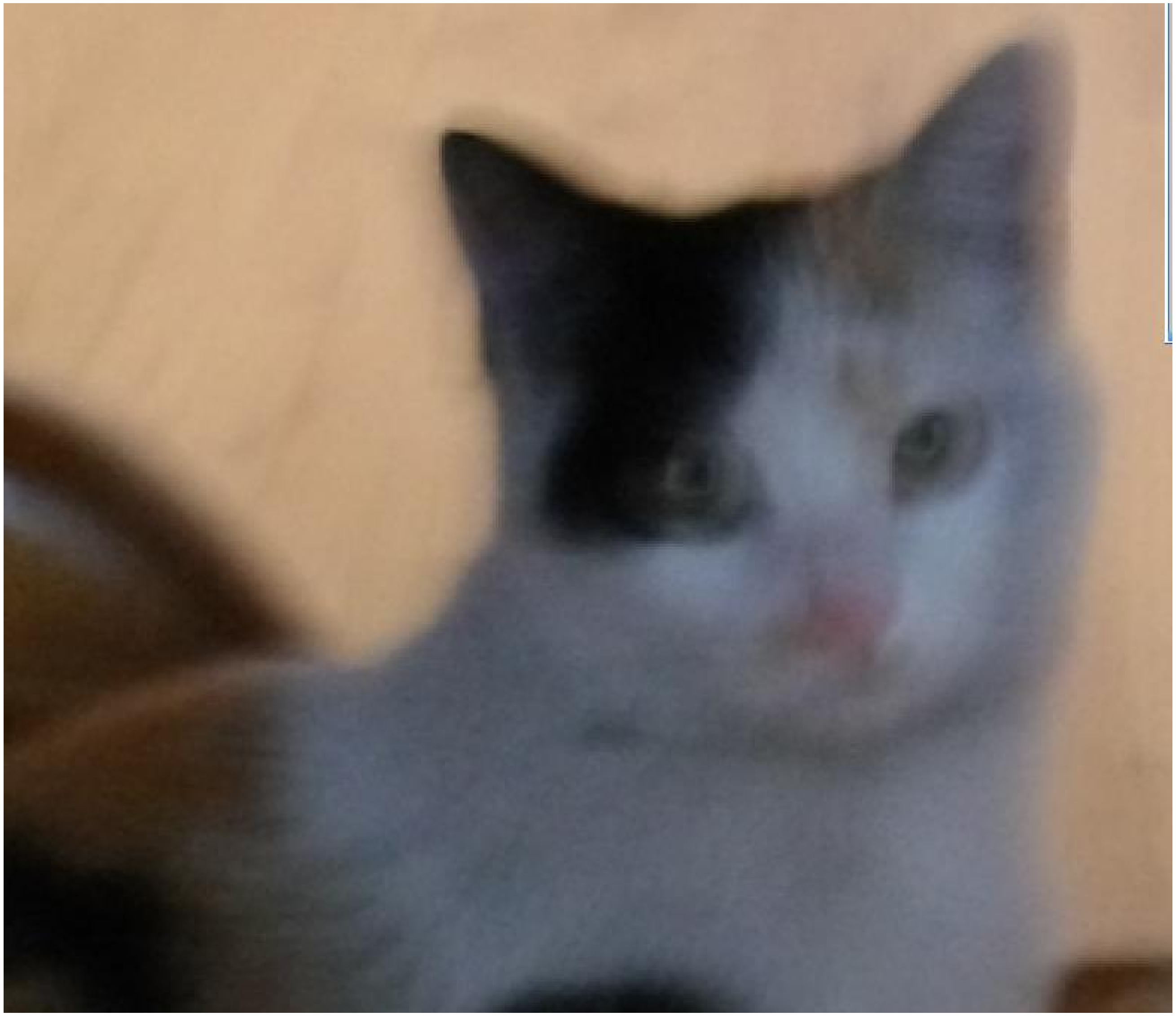


Fig. 1. Experiment 1. Left: Hubble data after despiking. Right: restored by our algorithm.

Geman & Yang

Would you please deblur me?



Challenge: unknown B

Now, in many cases B is not given. How do we find it?

One idea: we can set up a Gibbs sampler to repeatedly draw from

$$p(B|F,G), \quad p(F|B,G)$$

(the latter maybe by FFT approach)

$p(B|F,G)$ can be obtained using multivariate Gaussian, however a lot of computation when B is large

Will this work?

Can we use FFT for $p(B|F,G)$ somehow?

References

S. Geman and D. Geman (1984). Stochastic relaxation, Gibbs distributions and the Bayesian restoration of images. *IEEE Trans. Pattern Anal. Mach. Intell.*, **6**, 721-741.

see

`http://cis.jhu.edu/people/faculty/geman/publications.html`

D. Geman and C. Yang (1995). Nonlinear image recovery with half-quadratic regularization *IEEE Transactions on Image Processing*, **4**, no. 7, 932- 945

J.G. Nagy and D.P.O'Leary (2003). Image Deblurring: I can see clearly now. *Computing in Science and Engineering*, **5**, no. 3, 82-84

Thanks!