KEY

Midterm Exam
Math 586 Fall 2011

Problem	1	2	3	4	5	6	7	Total	Grade
Earned									
Possible	7	7	6	7	7	8	8	50	

1. For two observations, X_1 and X_2, with variance 1 each, denote their average as $\overline{X}$. Find $Var(\overline{X})$ when

(a) X_1 and X_2 are independent of each other

$Var(\overline{X}) = Var\left(\frac{X_1+X_2}{2}\right) = \frac{1}{4}\left[Var(X_1)+Var(X_2)\right] = 1/2$

(b) Correlation coefficient r between X_1 and X_2 is 0.8

$Var\left(\frac{1}{2}X_1 + \frac{1}{2}X_2\right) = \begin{bmatrix}\frac12 & \frac12\end{bmatrix}\begin{bmatrix}1 & 0.8\\ 0.8 & 1\end{bmatrix}\begin{bmatrix}1/2\\ 1/2\end{bmatrix}$

$\Sigma = Cov\left(\begin{bmatrix}X_1\\ X_2\end{bmatrix}\right)$

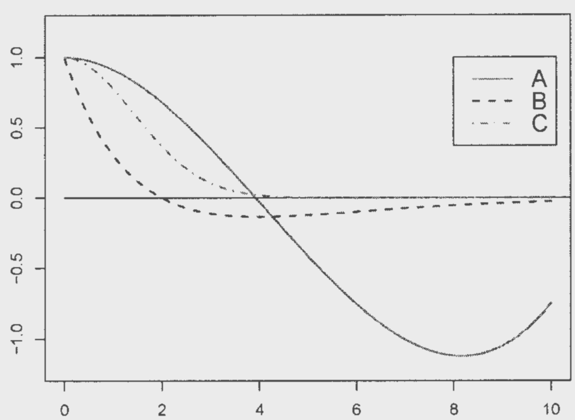
$= \frac{1}{4}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1 & 0.8\\ 0.8 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1.8\\ 1.8\end{bmatrix}$

$= \frac{3.6}{4} = 0.9$

2. (a) Which of the following graphs represent allowable **covariance** models? Explain.

A is not allowable
$(Cov(V(x) - V(x+8))$
$\simeq -1.1$
cannot be higher than
$Var(V(x)) = 1$



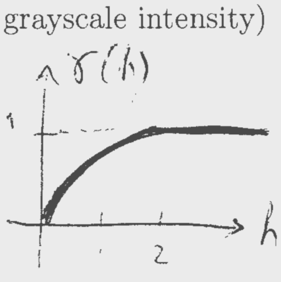
(b) Sketch an example of each

(i) a variogram model without nugget, with sill of 1 and practical range of 2.

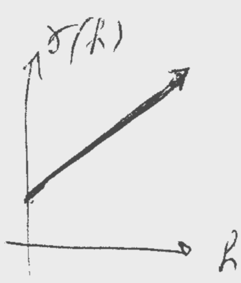
(ii) a variogram model with a nugget and no sill

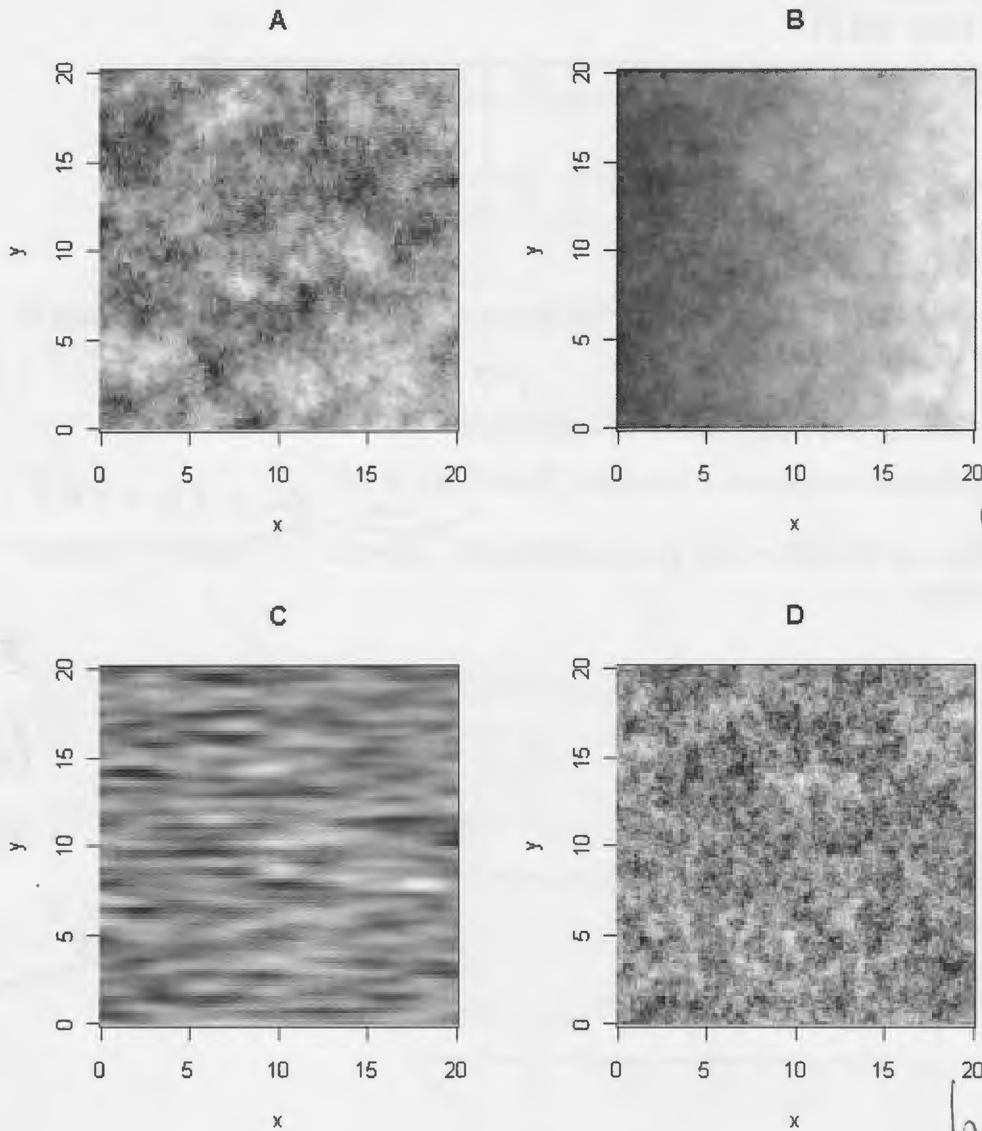
3. Which of the following 2-dimensional random fields (the value of V is given by grayscale intensity)

(i)



(ii)





4.a)
Unbiased: need
$\mathbb{E}(V_0) = \mathbb{E}(\hat{V}_0)$,
$\hat{V}_0 = \sum_{j=1}^{n} \lambda_j V_j$.
We assume
$\mathbb{E}(V_k) = m$,
m is known.
Then let
$\boxed{\tilde{V}_k = V_k - m}$
$\tilde{V}_k$ have mean 0.
λ_1
$\mathbb{E}(\hat{V}_0) = \sum \lambda_j \mathbb{E}(\tilde{V}_j) = 0$

(a) has Gaussian variogram C (it's smooth)

(b) Is not stationary B (it contains a trend; increasing from left to right)

(c) Is not isotropic C (it's "stretched" horizontally)

4. (a) Explain how in Simple Kriging, the unbiasedness condition is enforced.

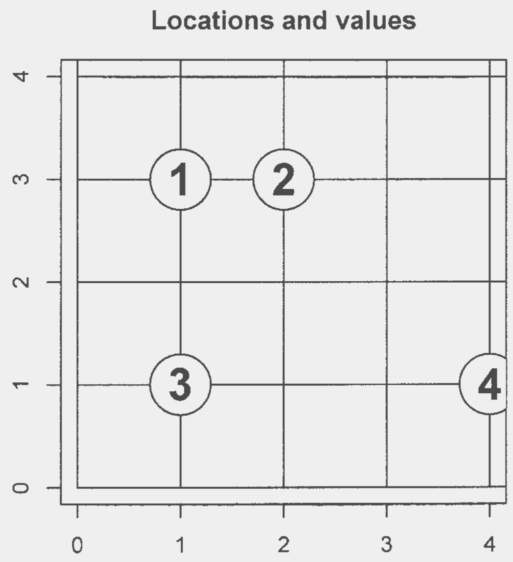
(b) Give an example of a process that has a variogram with no sill. Describe the way the process is constructed. W_k: discrete Brownian motion

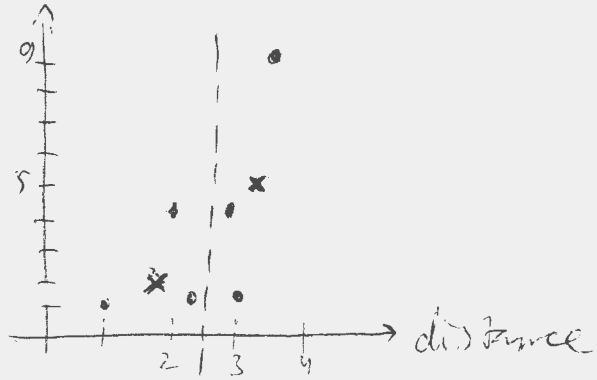
5. For the data (locations and values shown) plot the variogram cloud and compute the empirical variogram for distance classes (bins) (0, 2.5] and (2.5, 5]

let $V_0, V_1, V_2, V_3, \ldots$ be white noise (independent) with variance σ^2.

Let $W_0 = V_0$, $W_1 = V_0 + V_1$, $\ldots$, $W_k = V_0 + V_1 + \ldots + V_k$

then $\mathrm{Var}(W_k) = (k+1)\sigma^2$, so W_k has a linear variogram





pair	distance	sq. diff	
(1,2)	1	1	
(1,3)	2	4	(1.75, 2)
(2,3)	$\sqrt{5}$	1	
(2,4)	$\sqrt{8}$	4	
(3,4)	3	1	(3.14, 4.67)
(1,4)	$\sqrt{13}$	9	

6. The vector $\mathbf{X} = (X_1, X_2, X_3)'$ has multivariate normal distribution with mean $\mathbf{0}$ and covariance matrix

$$\Sigma = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

a) $\begin{bmatrix} a_2 \\ a_3 \end{bmatrix}' = \underbrace{[1 \ -1]}_{\Sigma_{12}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}}_{\Sigma_{22}^{-1}} = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}'$

(a) Compute the *best linear unbiased estimate* (BLUE) $\hat{X}_1$ of X_1 given X_2, X_3. That is, find the constants a_2, a_3 such that

$$\hat{X}_1 = a_2 X_2 + a_3 X_3$$

Also, find the mean square error (MSE) of this estimate.

$MSE = \sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$
$= 2 - [1 \ -1/2]\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
$= 2 - (1 + \frac{1}{2})$
$= 1/2$

(b) If, instead, we have found the BLUE $\tilde{X}_1$ of X_1 given X_2 only, will its MSE be higher or lower, compared to the part (a)? Explain.

(b) will be higher (since $a_2 = 1$, $a_3 = -1/2$ minimizes MSE)

7. (a) For the linear regression problem below, estimate coefficients β in the equation

$$\hat{y} = \beta_0 + \beta_1 x$$

x	y
−2	−1
−1	−1
0	0
1	1
2	1

$X = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$

$\hat{\beta} = (X'X)^{-1} X'\vec{y} = \begin{bmatrix} 1/5 & 0 \\ 0 & 1/10 \end{bmatrix}\begin{bmatrix} 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/5 \end{bmatrix}$

$X'X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$

$X'\vec{y} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$

(b) The standard deviation of residuals is 0.32 and the standard deviation of y is 1. Compute the correlation coefficient between x and y.

$R^2 = 1 - \frac{\sigma^2_{resid}}{\sigma^2_y} = 1 - \frac{0.32^2}{1^2} = 0.898$

$\Rightarrow R = \sqrt{0.898} = 0.947$