Lecture 7: Variograms

Math 586

The wide-sense stationarity (WSS) assumption can be restrictive - generalize?
Instead of covariance $C(\mathbf{h})$, look at the differences $V(\mathbf{x}+\mathbf{h}) - V(\mathbf{x})$. Their variance may exist even if $Var[V(\mathbf{x})]$ does not exist.

Definition
$V(\mathbf{x})$ is an **Intrinsic Random Field** of order 0 (IRF-0) if

1. $\mathbb{E}[V(\mathbf{x})] = m$ (constant)
2. $Var[V(\mathbf{x}+\mathbf{h}) - V(\mathbf{x})]$ is only a function of lag $\mathbf{h}$.

Definition
$\gamma(\mathbf{h}) = \dfrac{1}{2} Var[V(\mathbf{x}+\mathbf{h}) - V(\mathbf{x})]$ is called the **(semi)variogram** for $V(\mathbf{x})$.

Example: Discrete case

The following is a simple example of a non-stationary random field that is still IRF-0:
Consider $W_1, W_2, ..., W_n$ independent, $\mathbb{E}(W_i) = 0$, $Var(W_i) = \sigma^2$.
Let $V(0) = 0$,

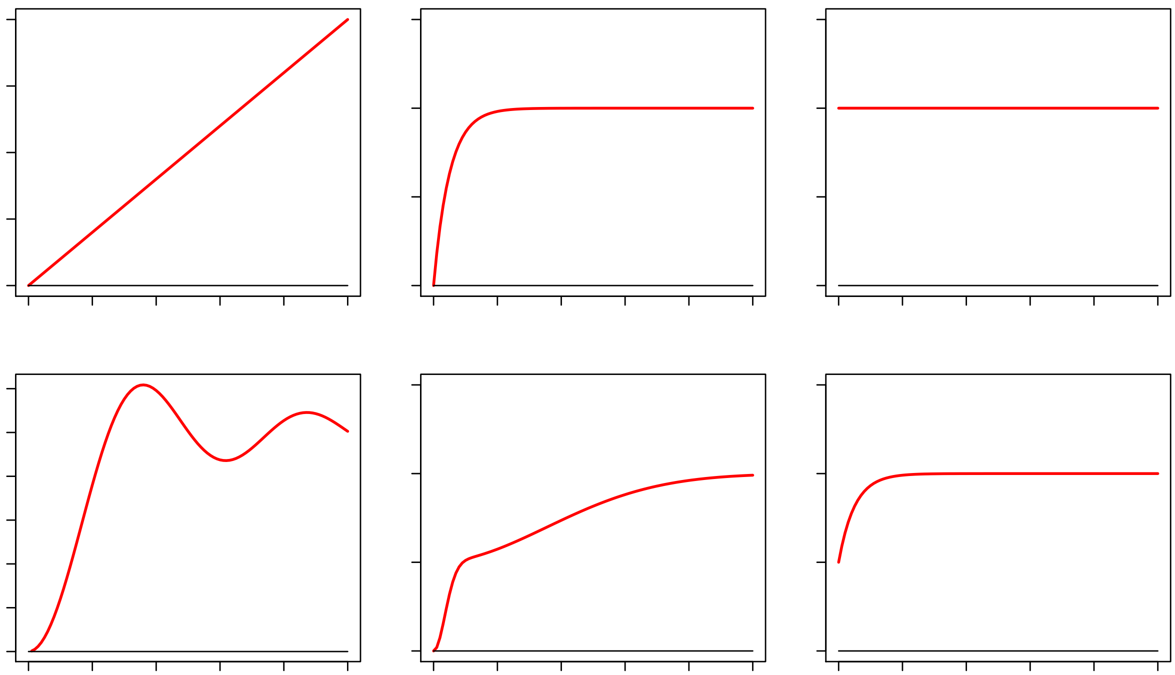
$$V(k) = \sum_{j=1}^{k} W_j.$$

Note that $Var[V(k)] = k\sigma^2$, therefore $V(k)$ is not WSS. (Why?)
But

$$V(k+j) - V(k) = \sum_{i=k+1}^{k+j} W_i \;\Rightarrow$$

$Var[V(k+j) - V(k)] = j\sigma^2$ only depends on j, so V is IRF-0.
This V is called *random walk*.

Some variogram shapes:



Relation between variograms and covariances:
In case of WSS $V(\mathbf{x})$, with covariance function $C_V(\mathbf{h})$, what is $\gamma_V(\mathbf{h})$?

$$\gamma(\mathbf{h}) = \frac{1}{2} Var[V(\mathbf{x}+\mathbf{h}) - V(\mathbf{x})] =$$

$$= \frac{1}{2} \{Var[V(\mathbf{x}+\mathbf{h})] + Var[V(\mathbf{x})] - 2Cov[V(\mathbf{x}+\mathbf{h}), V(\mathbf{x})]\} =$$

$$= \frac{1}{2}[C_V(0) + C_V(0) - 2C_V(\mathbf{h})] = C_V(0) - C_V(\mathbf{h})$$

So, for a WSS process, the variogram is just the upside-down covariance.

As $|\mathbf{h}| \to \infty$, $\gamma_V(\mathbf{h}) \to C_V(0) = Var[V(\mathbf{x})]$, also called **the sill**.

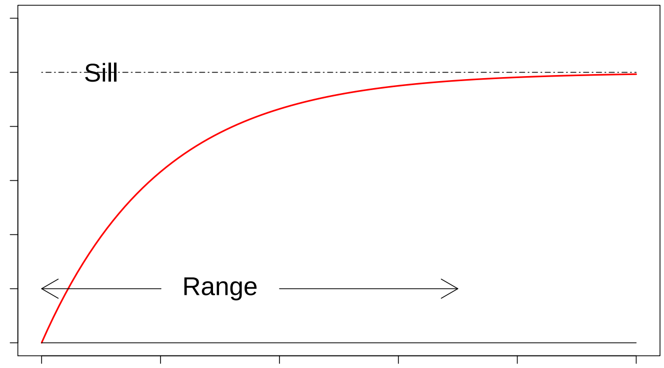
If $V(\mathbf{h})$ is stat. isotropic, and $\ell =$ smallest such that $\gamma_V(\ell) = C_V(0)$ then ℓ is **the range**.

Also (loosely) say that *practical range* $\approx$ the distance beyond which covariance is negligible.

Correlation Scale =

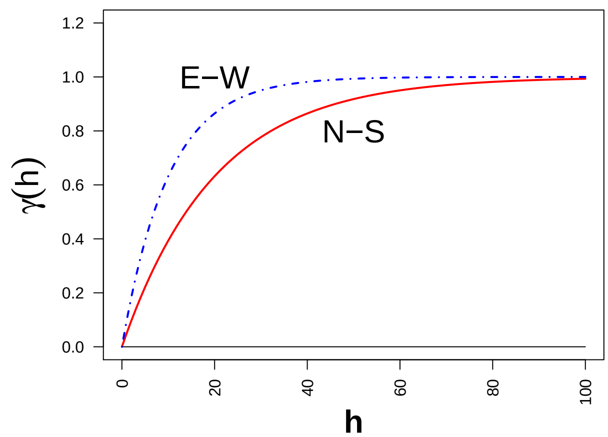
λ such that $\dfrac{C(\lambda)}{C(0)} \approx e^{-1}$

Practical range and correlation scale are not rigorously defined.



Non-isotropic case could have several scales/ranges depending on direction.

Ror WSS case, only one sill, however. (Why?)



Variogram properties

(i.) $\gamma(\mathbf{h}) \geq 0$ (Why?)

(ii.) $\gamma(-\mathbf{h}) = \gamma(\mathbf{h})$ (symmetry)

(iii.) $-\gamma(\mathbf{h})$ is a positive-semidefinite function (not just any function will work!)

(iv.) Behavior at ∞:

$$\lim_{|\mathbf{h}|\to\infty} \frac{\gamma(|\mathbf{h}|)}{|\mathbf{h}|^2} = 0$$

i.e. bounded by quadratic.
If the estimated variogram looks quadratic $\Rightarrow$ trend not removed (i.e. $\mathbb{E}\left[V(\mathbf{x})\right]$ not constant).

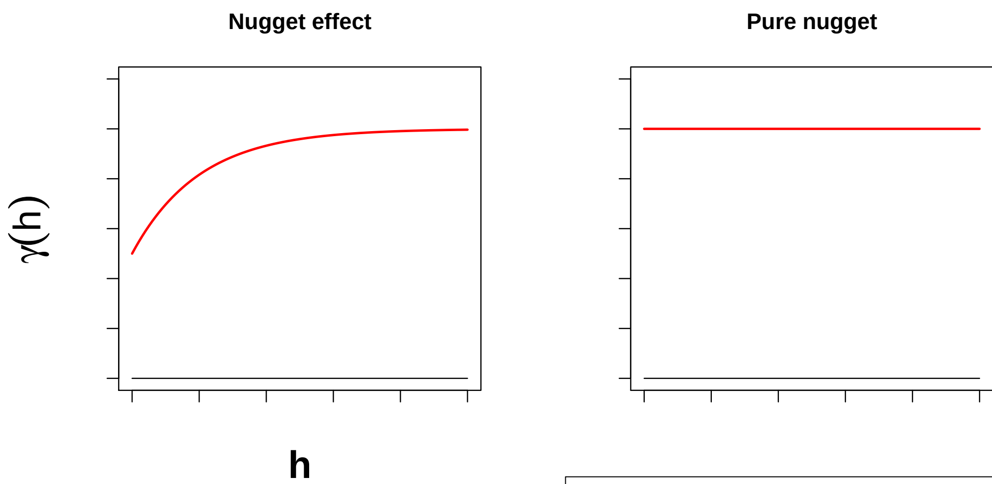
(v.) Behavior near 0:

(a) $\gamma(\mathbf{h}) \sim A|\mathbf{h}|^2$ near 0 $\Rightarrow$ $V(\mathbf{x})$ is smooth (has a derivative in mean-square sense)

(b) $\gamma(\mathbf{h}) \sim A|\mathbf{h}| \;\Rightarrow\; V(\mathbf{x})$ is continuous but not differentiable

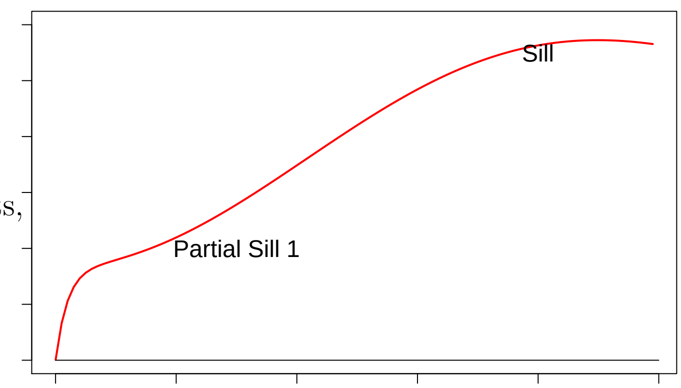
(c) $\gamma(\mathbf{h})$ is discontinuous at 0 $\;\Rightarrow\;$ "nugget" effect.

Nugget: (independent) measurement error or fine-scale variation. Estimated variograms often show nuggets. Pure nugget: independent values.



(vi.) Multi-Scale variograms:

Both short- and long-range effects, e.g. stream deposition: local (short-range) and global (long-range) processes.



Some functional forms for variograms:

Let $h = |\mathbf{h}|$.

- Power (linear when $c = 1$; has no sill):

$$\gamma(\mathbf{h}) = Ah^c, \quad 1 \leq c < 2$$

- Exponential (A = "sill", B = "scale")

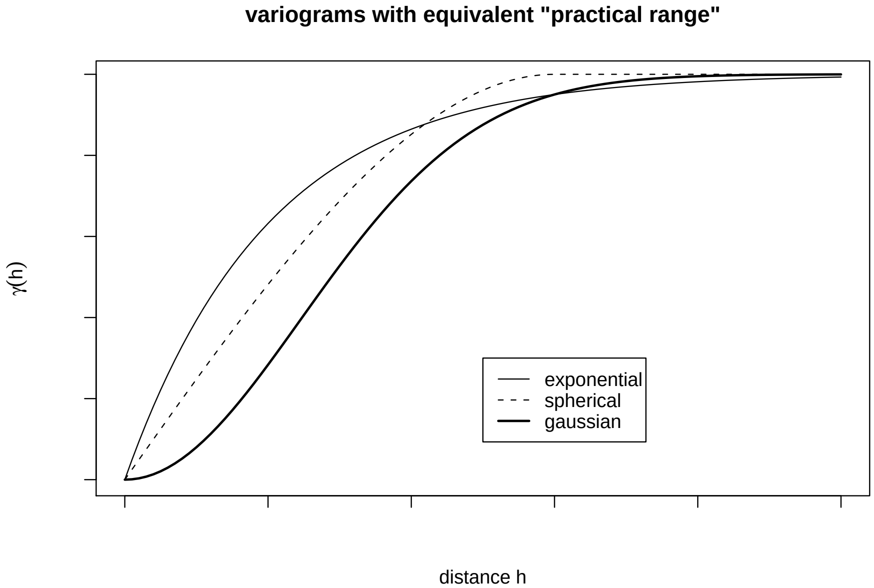
$$\gamma(\mathbf{h}) = A\left[1 - e^{-h/B}\right]$$

- Spherical

$$\gamma(\mathbf{h}) = \begin{cases} A\left[1.5(h/B) - 0.5(h/B)^3\right] & \texttt{if} \quad h < B \\ A & \texttt{otherwise} \end{cases}$$

- Gaussian

$$\gamma(\mathbf{h}) = A\left[1 - e^{-(h/B)^2}\right]$$

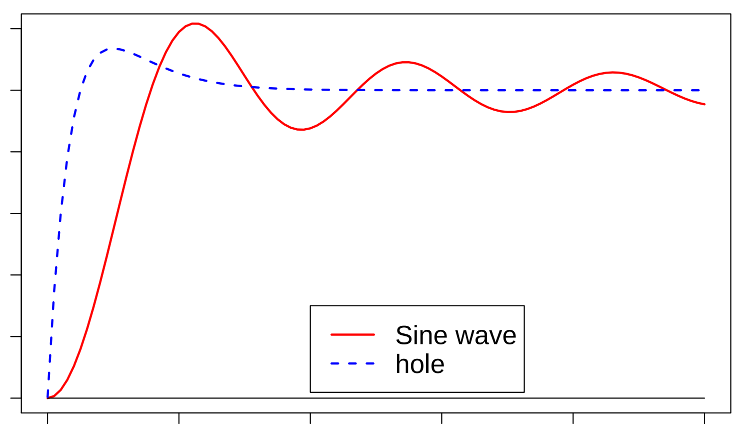


- "Hole" variogram (models layering, valid for 1d only)

$$\gamma(h) = A\left[1 - (1 - h/B)e^{-h/B}\right]$$

- Sine wave (models layering, periodicities)

$$\gamma(\mathbf{h}) = A[1 - (B/h)\sin(h/B)]$$



- Nested models

  (a.) Consider r.f. $V(\mathbf{x})$ with variogram $\gamma_V(\mathbf{h})$, and $\varepsilon(\mathbf{x})$ independent of $V(\mathbf{x})$, mean 0, variance σ^2, and $\varepsilon(\mathbf{x}_1)$ independent of $\varepsilon(\mathbf{x}_2)$ for $\mathbf{x}_1 \neq \mathbf{x}_2$. (Models observational error, or pure nugget.)

  Let $W(\mathbf{x}) = V(\mathbf{x}) + \varepsilon(\mathbf{x})$.
  $\gamma_W(\mathbf{h}) = ?$

  $$\gamma_W(\mathbf{h}) = \frac{1}{2}\mathbb{E}\left[W(\mathbf{x}+\mathbf{h}) - W(\mathbf{x})\right]^2 = \frac{1}{2}\mathbb{E}\left[V(\mathbf{x}+\mathbf{h}) - V(\mathbf{x}) \;+\; \varepsilon(\mathbf{x}+\mathbf{h}) - \varepsilon(\mathbf{x})\right]^2 =$$

  $$= \frac{1}{2}\mathbb{E}\left[V(\mathbf{x}+\mathbf{h}) - V(\mathbf{x})\right]^2 \;+\; \frac{1}{2}\mathbb{E}\left[\varepsilon(\mathbf{x}+\mathbf{h}) - \varepsilon(\mathbf{x})\right]^2 +$$

  $$+\mathbb{E}\left\{\left[V(\mathbf{x}+\mathbf{h}) - V(\mathbf{x})\right]\left[\varepsilon(\mathbf{x}+\mathbf{h}) - \varepsilon(\mathbf{x})\right]\right\} =$$

  $$= \gamma_V(\mathbf{h}) + \sigma^2 \;\Rightarrow\; \text{Nugget effect}$$

  (b.) Generalization:
  $V_1(\mathbf{x}), V_2(\mathbf{x}), ..., V_k(\mathbf{x})$ independent, with variograms $\gamma_1(\mathbf{h}), \gamma_2(\mathbf{h}), ..., \gamma_k(\mathbf{h})$.

If $W(\mathbf{x}) = V_1(\mathbf{x}) + V_2(\mathbf{x}) + ... + V_k(\mathbf{x})$ (independent), then $\gamma_W(\mathbf{h}) = \gamma_1(\mathbf{h}) + \gamma_2(\mathbf{h}) + ... + \gamma_k(\mathbf{h})$.

Misspecification Example (1-d case):

Let $V(x)$ IRF-0
$W(x) = V(x) + \beta x$. Is W an IRF-0?
Suppose blindly calculate $\frac{1}{2}\mathbb{E}\left[W(x+h) - W(x)\right]^2 =$

$$= \frac{1}{2}\mathbb{E}\left[V(x+h) - V(x) \;+\; \beta(x+h) - \beta x\right]^2 =$$

$$= \frac{1}{2}\mathbb{E}\left[V(x+h) - V(x)\right]^2 \;+\; \frac{\beta^2 h^2}{2}$$

Thus, the new variogram is $\gamma_V(h) + \frac{\beta^2 h^2}{2}$ (trend effect).

Note that W is not IRF-0: should detrend first!