Lecture 6: Stochastic Processes (Random Fields)

Math 586

So far, we had deterministic mean values (trend) with independent random fluctuations around the mean.

However, we are also interested in cases where we have continuity/"integrity" of measurement, for example:

(i) Porosity = function of location
(ii) Ore grade = function of location
(iii) Temperature = function of location and time

Use of random process models:

- study correlation/variability
- Predict/interpolate

Aliases: *Random field (r.f.) = Spatial Stochastic Process = Regionalized variable*

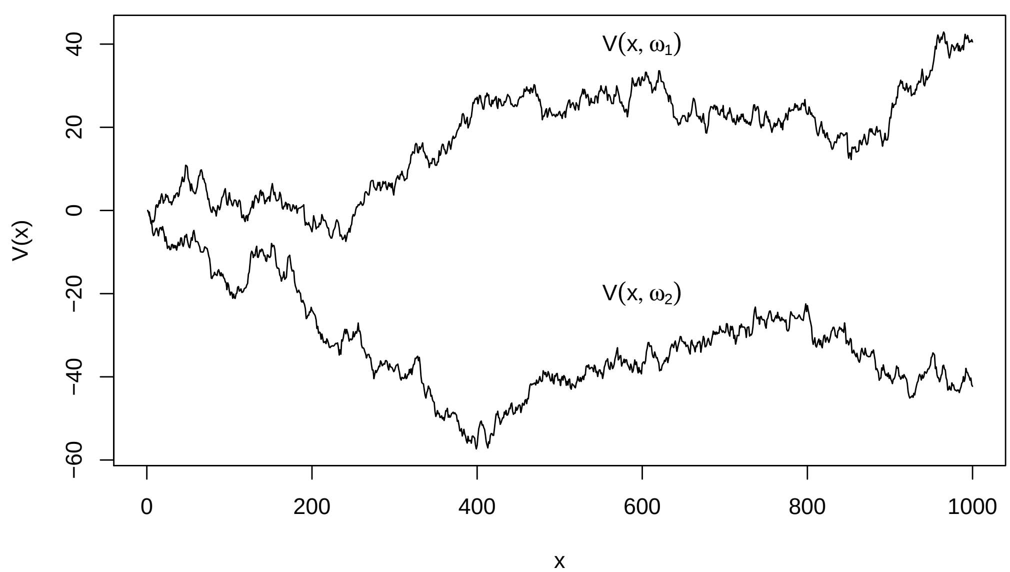
> **Definition**
> $V(\mathbf{x}), \mathbf{x} \in \mathbb{R}^m$ is a **random field** if for each fixed $\mathbf{x}_0$, $V(\mathbf{x}_0)$ is a random variable. [does not say much]

$V(\mathbf{x};\omega)$ = a realization ($\omega \leftrightarrow$ randomness), a random function of $\mathbf{x}$.
In practice, we often observe the process only once:

$$v_1(\mathbf{x}) = V(\mathbf{x};\omega_1)$$

(e.g. Earth's climate might be modeled by a stochastic process, but we only see one realization of it!)

Two sample paths of a random process



> **Definition (probability description)**
> $V(\mathbf{x})$ is completely described in probabilistic sense if
> for any n, for all $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n$ the joint distribution of $V(\mathbf{x}_1), V(\mathbf{x}_2), ..., V(\mathbf{x}_n)$ is known.

A lot of information! (For example, we would know means and correlations between $V(\mathbf{x}_i)$ and $V(\mathbf{x}_j)$, but so much more!)

Important case: Gaussian random field when $V(\mathbf{x}_1), V(\mathbf{x}_2), ..., V(\mathbf{x}_n)$ are multivariate normal.
For Gaussian r.f. the following are enough:
(i) mean function $m(\mathbf{x}) = \mathbb{E}[V(\mathbf{x})]$
(ii) variance/covariance function $C(\mathbf{x}, \mathbf{y}) = Cov[V(\mathbf{x}), V(\mathbf{y})]$
Still a lot of info, but much less than before. Still produces a lot of useful results.

Further restriction: consider constant mean $m(\mathbf{x}) = m$ and covariance depending only on **lag** (separation) $\mathbf{h} = \mathbf{x} - \mathbf{y}$.

> **Definition**
> R.f. $V(\mathbf{x})$ is **second-order stationary** (statistically homogeneous) if
> 1. $\mathbb{E}[V(\mathbf{x})] = m$ (constant)
> 2. $Cov[V(\mathbf{x}), V(\mathbf{y})] \equiv C(\mathbf{x} - \mathbf{y})$ only a function of lag $\mathbf{x} - \mathbf{y}$.

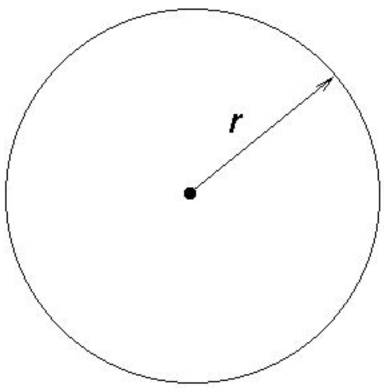
$C(\mathbf{h})$ = autocovariance.

Further restriction:

Let $|\mathbf{h}| = \sqrt{h_1^2 + h_2^2 + ... + h_n^2}$ euclidean norm.

R.f. is **statistically isotropic** if

$$C(\mathbf{h}) = C(|\mathbf{h}|), \text{ i.e. a function of only a distance}$$



$C(r)$ tells us how the points distance r apart are related.
Possible 1-d covariances: