(1.) (a) $\hat{\lambda}_1 = \bar{X}$, $\hat{\lambda}_2 = \bar{Y}$; $se(\hat{\lambda}_1) = \frac{\sigma_1}{\sqrt{n}}$, $\hat{se}(\hat{\lambda}_1) = \frac{\sqrt{\bar{X}}}{\sqrt{n}}$; $\hat{se}(\hat{\lambda}_2) = \frac{\sqrt{\bar{Y}}}{\sqrt{n}}$

$\Rightarrow$ 95% C.I. is

$$(\bar{X} - \bar{Y}) \pm z_{\alpha/2}\sqrt{\frac{\bar{X}}{n} + \frac{\bar{Y}}{n}}$$

$$7.3 - 5.1 \pm 1.96\sqrt{\frac{7.3}{25} + \frac{5.1}{25}}$$

$$2.2 \pm 1.4$$

(b) no, $\lambda_1 \neq \lambda_2$ because $0 \notin$ C.I.

(c) $$W = \frac{\bar{X} - \bar{Y}}{\sqrt{\bar{X}/n + \bar{Y}/n}} = \frac{7.3 - 5.1}{\sqrt{7.3/25 + 5.1/25}} \simeq 3.12$$

p-value $= 2\Phi(-3.12) \simeq 0.0018$

(2.) a) $L(\theta) = \prod \frac{1}{\theta} e^{-x_i/\theta}$, $\ln L(\theta) = \sum\left(-\ln\theta - \frac{x_i}{\theta}\right)$
$= -n\ln\theta - \frac{\sum x_i}{\theta}$

$(\ln L(\theta))' = -\frac{n}{\theta} + \frac{\sum x_i}{\theta^2} = 0$

$\Rightarrow \hat{\theta} = \frac{\sum x_i}{n} = \bar{x}$

b) $\ln f(\theta) = -\ln\theta - \frac{x}{\theta}$, $(\ln f)' = -\frac{1}{\theta} + \frac{x}{\theta^2}$

$(\ln f)'' = \frac{1}{\theta^2} - \frac{2x}{\theta^3}$, $I(\theta) = -E((\ln f)'') = E\left(-\frac{1}{\theta^2} + \frac{2x}{\theta^3}\right)$

$= \frac{1}{\theta^2}$ (as $E(X) = \theta$)

c) $\bar{x} \pm (nI(\theta))^{-1/2} z_{\alpha/2} \Rightarrow \bar{x} \pm z_{\alpha/2}\frac{\bar{x}}{\sqrt{n}}$, or

$41.32 \pm 1.96\frac{41.32}{\sqrt{100}}$

41.32 ± 8.10

d) $\lambda = 2\ln\left(\frac{L(\hat{\theta})}{L(\theta_0)}\right)$, $\lambda = 2\left[-n\ln(\hat{\theta}) - \frac{\sum x_i}{\hat{\theta}} + n\ln(\theta_0) + \frac{\sum x_i}{\theta_0}\right]$

$= 2\left[-100\ln(41.3) - \frac{4132}{41.32} + 100\ln(40) + \frac{4132}{40}\right] \simeq 0.203$

$\lambda < \lambda^* = $ qchisq(0.95, df = 1) $= 3.84 \Rightarrow$ Accept H_0

3. a) $V(\bar{X}) = \frac{\sigma^2}{n} = \frac{2}{3}$

b) $V(\bar{X}) = \frac{1}{9} V(X_1 + X_2 + X_3) =$

$$= \frac{1}{9}\left[2+2+2+\underset{\substack{\uparrow \\ \#\ \text{terms}}}{3\times 2}\ \mathrm{Cov}(X_i, X_j)\right],$$

$$\boxed{\begin{aligned}\mathrm{cov}(X_i, X_j) &= \\ &= \sigma_i \sigma_j \,\mathrm{cor}(\cdot,\cdot) \\ &= -1\end{aligned}}$$

$$= \frac{1}{9}\left[6 + 6(-1)\right] = 0 \qquad (!)$$

4. a) median for Exp. distrib. is:

solve $F(m) = \frac{1}{2} \Rightarrow 1 - e^{-m/\beta} = \frac{1}{2}$

$\Rightarrow -\frac{m}{\beta} = \ln(1/2) \Rightarrow m = \beta \ln 2$

$\Rightarrow$ MLE(median) $= \hat{\beta} \ln 2 = 2.35 (\ln 2)$.

b) i. not regular: $f_\theta(x) = \begin{cases} \frac{1}{2\theta}, & -\theta < x < \theta \\ 0, & \text{else.} \end{cases}$

$\frac{\partial f}{\partial \theta}$ does not exist for ~~$x = \pm\theta$~~ $\theta = \pm x$

ii. $f_\theta(x) = \frac{1}{\theta} e^{-x/\theta}, x > 0$

regular: $\frac{\partial f}{\partial \theta}, \frac{\partial^2 f}{\partial \theta^2}$, Fisher info etc.
they all exist

c) $X_1, \ldots, X_n \sim$ Uniform $[-\theta, \theta]$

$\Rightarrow$ need $\max(X_i) \leq \theta, \quad \min(X_i) \geq -\theta$

$\Rightarrow \theta = \max_{i=1,\ldots,n}\{ \cancel{\max} |X_i| \}$

5. a) $\mathcal{N}(\bar{x}, \bar{\sigma}^2)$, where $\bar{x} = \dfrac{\frac{X}{2} + \frac{0}{5}}{\frac{1}{2} + \frac{1}{5}} = \frac{5}{7}$

$$\bar{\sigma}^2 = \frac{1}{\frac{1}{2} + \frac{1}{5}} = \frac{10}{7}$$

b) prior
$f(p) \sim$ Uniform $[0,1]$ = Beta $(1, 1)$

$\Rightarrow$ posterior is Beta $(X+1, n-X+1)$

$= \text{Beta}(\underset{\alpha}{4}, \underset{\beta}{8})$

post. mean $= \dfrac{\alpha}{\alpha+\beta} = \dfrac{4}{4+8} = 2/3$.

c) prior $f(\theta) = \frac{c}{\theta}$, likelihood $f(X|\theta) = \theta e^{-\theta X}$

$\Rightarrow$ posterior $\propto f(\theta) f(X|\theta) = \frac{c}{\theta}(\theta e^{-\theta X}) = c e^{-\theta X}$

$\Rightarrow$ Exponential with mean $\frac{1}{X}$.