Midterm Exam

Name (please print) KEY

Math 483, Fall 2017 October 19, 2017

Show work and correct notation for full credit. Give numerical or simple fraction answers whenever possible.

Problem	1	2	3	4	5	6	Total
Earned							
Possible	10	10	10	10	10	10	50

1. (a) $X_n \xrightarrow{P} 0$:
$P(|X_n - 0| > \varepsilon) = P(X_n > \varepsilon) = 1 - F_{X_n}(\varepsilon) = e^{-n\varepsilon} \to 0$

Solve 5 problems out of 6. Cross one out.

1. (a) Does the sequence $X_n \sim$ `Exponential`$(mean = 1/n)$ converge in probability? If no, explain why not. If yes, find the limit and explain why the sequence converges to it.

(b) Give an example of something converging in distribution to a non-trivial distribution (i.e. not a point mass).

(b) CLT $\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{D} N(0,1)$, many others possible.

2. Let $X_1, ..., X_n$ be an i.i.d. sample from Poisson distribution with parameter (mean) θ, with $n = 45$ and $\bar{X} = 36.6$. Find a **92%** C.I. for θ.

$V(X_i) = \theta$, $\hat{\theta} = \bar{X}$

$\Rightarrow$ C.I. is $\bar{X} \pm z_{\alpha/2} \frac{\sqrt{\bar{X}}}{\sqrt{n}}$, or $36.6 \pm 1.75 \sqrt{\frac{36.6}{45}}$, 36.6 ± 1.58,

$[35.0 ; 38.2]$

3. Let $X_i \sim N(0, var = 5)$. Find $Var(X_1 - X_2)$ assuming:

(a) X_i are independent $= Var(X_1) + Var(X_2) = 10$

(b) $corr(X_1, X_2) = 0.5$ $= Var(X_1) + Var(X_2) - 2\underbrace{Cov(X_1, X_2)}_{= \sigma_1 \sigma_2 corr(X_1, X_2)}$

$= 5 + 5 - 2 \times \sqrt{5}\sqrt{5}\, 0.5 = 5$

4. (a) Explain how CLT implies that Gamma distribution with "large" α is approximately Normal.

$(n = \alpha)$ independent summands
$X = Y_1 + \ldots + Y_n$
where $Y_i \sim Expon(\beta)$
so $X = \bar{Y} n$, and $\bar{Y}$ is approx. Normal $(\mu, \frac{\sigma^2}{n})$, $\mu = \beta$, $\sigma^2 = \beta^2$ for Expon.(β)

(b) Write the normal approximation for $X \sim Gamma(\alpha = 100, \beta = 1)$

$X \sim N(n\mu, n\sigma^2) = N(\alpha\beta, Var = \alpha\beta^2) = N(100, Var = 100)$

(c) Use Delta-method to find the approximate distribution for $\sqrt{X}$.

$g(x) = \sqrt{x}$ $\sim N(\sqrt{100}, [g'(100)]^2 \times 100) = N(10, Var = \frac{1}{4})$

$g'(x) = \frac{1}{2\sqrt{x}}$, $g'(100) = \frac{1}{20}$

(a) $\bar{X}_1 - \bar{X}_2 \pm z_{\alpha/2}\sqrt{\frac{S_1^2}{n_1}+\frac{S_2^2}{n_2}}$ is $50.556 - 48.372 \pm 2.33\sqrt{\frac{6.872^2}{55}+\frac{7.039^2}{40}}$

$= 2.184 \pm 3.374$

(in \$1,000)

or

$[-1.19, 5.56]$

5. (a) Two random samples of graduates from two different universities were compared, and their average starting salaries were as follows

Sample	Size	Mean	St.Dev.
1	55	\$50,556	\$6,872
2	40	\$48,372	\$7,039

Obtain a 98% confidence interval for the difference between the "true" starting salaries of the two universities.

(b)

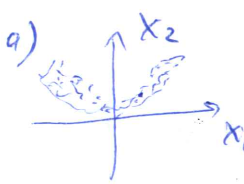
$$\kappa(a+bX) = \frac{\mathbb{E}\left[(a+bX-\mathbb{E}(a+bX))^3\right]}{\left(b^2 Var(X)\right)^{3/2}} = \frac{\mathbb{E}\left[(a+bX-a-b\mathbb{E}(X))^3\right]}{b^3\, Var(X)^{3/2}} = \frac{b^3\,\mathbb{E}\left[(X-\mathbb{E}X)^3\right]}{b^3\left(Var(X)\right)^{3/2}} = \kappa(X).$$

(b) The *skewness* is defined as

$$\kappa = \frac{\mathbb{E}\left[(X - \mathbb{E}\,X)^3\right]}{[Var(X)]^{3/2}}$$

It measures the degree of asymmetry in the distribution. Normal distribution, for example, has $\kappa = 0$.
Show that for any a and b, $\kappa(a + bX) = \kappa(X)$.

6. True or False? Explanations are not necessary, but they won't hurt.

a) 

F (a) If the correlation coefficient between two random variables X_1 and X_2 is 0, then they must be independent.

T (b) $\tilde{\sigma} = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n+1}}$ is a consistent estimate of standard devation σ.

b) $\tilde{\sigma} = \sqrt{\frac{n-1}{n+1}}\, S$

consistent

(c) Let $\hat{p}_n$ be the sample proportion based on a sample of 0/1 outcomes $X_1, \ldots, X_n$, with $p = P(X_i = 1)$, $0 < p < 1$.
Then $\hat{p}_n$ is an asymptotically normal estimate of p. T

$\frac{\hat{p}_n - p}{\sqrt{p(1-p)/n}} \xrightarrow{D} N(0,1)$

By CLT

median is more resistant

F (d) Sample mean is more resistant to outliers (unusually high or low observations) than the sample median.

F (e) If $\hat{\theta}_n$ is an unbiased estimate of θ, then $\hat{\theta}_n^2$ is an unbiased estimate of θ^2.

if $\mathbb{E}(\hat{\theta}_n) = \theta$, $\mathbb{E}(g(\hat{\theta}_n)) = g(\mathbb{E}\hat{\theta}_n)$ only for g a linear function