Hw 8 Key

M483

(3.) a) $\hat{\mu} = \bar{X}$, $\hat{\sigma} = \sqrt{\dfrac{\Sigma (X_i - \bar{X})^2}{n}}$

$\tau = \mu + 1.645\,\sigma$ is 95^{th} percentile

$\Rightarrow$ (equivariance) $\hat{\tau} = \hat{\mu} + 1.645\,\hat{\sigma}$

b) $\nabla g = \begin{bmatrix} 1 \\ 1.645 \end{bmatrix} \Rightarrow se(\hat{\tau}) = \sqrt{V(\hat{\tau})} = \sqrt{\nabla g^T J_n \nabla g} =$

$= \sqrt{V(\hat{\mu}) + (1.645)^2 V(\hat{\sigma})} = \sqrt{\dfrac{\hat{\sigma}^2}{n} + (1.645)^2 \dfrac{\hat{\sigma}^2}{2n}}$

$\simeq 1.534 \dfrac{\hat{\sigma}}{\sqrt{n}}$. $\Rightarrow$ C.I. is $\hat{\tau} \pm z_{\alpha/2}\left(1.534 \dfrac{\hat{\sigma}}{\sqrt{n}}\right)$

(c) $\hat{\tau} \simeq 4.12$, $\hat{se}(\hat{\tau}) \simeq 0.55$

(7.) $\vec{\theta} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$, $\psi = p_1 - p_2$

a) $L(\vec{\theta}) = \binom{n_1}{X_1} p_1^{X_1} (1-p_1)^{n_1 - X_1} \binom{n_2}{X_2} p_2^{X_2} (1-p_2)^{n_2 - X_2}$

$\ln L(\vec{\theta}) = \text{const} + X_1 \ln p_1 + (n_1 - X_1) \ln (1-p_1)$
$\qquad + X_2 \ln p_2 + (n_2 - X_2) \ln(1-p_2)$

$\dfrac{\partial \ln L}{\partial p_1} = \dfrac{X_1}{p_1} - \dfrac{n_1 - X_1}{(1-p_1)} = 0 \Rightarrow X_1(1-p_1) = (n_1 - X_1) p_1$

$\Rightarrow \hat{p}_1 = \dfrac{X_1}{n_1}$; similarly $\hat{p}_2 = \dfrac{X_2}{n_2}$

$\Rightarrow \hat{\psi} = \hat{p}_1 - \hat{p}_2 = \dfrac{X_1}{n_1} - \dfrac{X_2}{n_2}$

b) $\ln f(\vec{\theta}) = \ln L(\vec{\theta})$ (here, only 1 sample each)

$\Rightarrow \dfrac{\partial^2 \ln f}{\partial p_1^2} = -\dfrac{X_1}{p_1^2} - \dfrac{(n_1 - X_1)}{(1-p_1)^2}$, $\dfrac{\partial^2 \ln f}{\partial p_1 \partial p_2} = 0$

$$\frac{\partial^2 \ln f}{\partial p_2^2} = -\frac{X_2}{p_2^2} - \frac{n_2 - X_2}{(1-p_2)^2}$$

$$\Rightarrow \mathbb{I}(\theta) = -\mathbb{E}\left(\frac{\partial^2 \ln f}{\partial p_i \, \partial p_j}\right)_{ij} = \begin{bmatrix} \frac{n_1}{p_1(1-p_1)} & 0 \\ 0 & \frac{n_2}{p_2(1-p_2)} \end{bmatrix}$$

$$\mathbb{E}\left(\frac{X_1}{p_1^2} + \frac{n_1 - X_1}{(1-p_1)^2}\right) = \frac{n_1 p_1}{p_1^2} + \frac{n_1 - n_1 p_1}{(1-p_1)^2}$$

$$= n_1\left(\frac{1}{p_1} + \frac{1}{1-p_1}\right) = n_1 \frac{1}{p_1(1-p_1)}$$

$J(\theta) = [\mathbb{I}(\theta)]^{-1}$

c) $$V(\hat{\psi}) = [1 \;\; -1] \begin{bmatrix} \frac{p_1(1-p_1)}{n_1} & 0 \\ 0 & \frac{p_2(1-p_2)}{n_2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} =$$

$$\nabla g(p_1, p_2) = \nabla(p_1 - p_2) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}$$

d)

$$\hat{p}_1 - \hat{p}_2 \pm 1.645\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

is $[-.009, 0.129]$

A. See Hw8key.r

I got $\text{eff}_N \simeq 1.47$, $\text{eff}_{T_3} \simeq 0.59$, $\text{eff}_{T_{10}} \simeq 1.25$

for this one, median is better