Hw 7 key

M 483

(2.) (a) Let $\hat{\mu} = \frac{\hat{a}+\hat{b}}{2} = \bar{X}$

$$\widehat{\sigma^2} = \frac{(\hat{b}-\hat{a})^2}{12} = \widehat{\sigma^2} = \frac{\sum (X_i - \bar{x})^2}{n}$$

$$\Rightarrow \begin{cases} \hat{b}+\hat{a} = 2\bar{x} \\ \hat{b}-\hat{a} = \sqrt{12\widehat{\sigma^2}} \end{cases} \Rightarrow \quad \hat{b} = \frac{2\bar{x}+\sqrt{12\hat{\sigma}^2}}{2} = \bar{x}+\sqrt{3}\,\hat{\sigma}$$

$$\hat{a} = \frac{2\bar{x}-\sqrt{12\hat{\sigma}^2}}{2} = \bar{x}-\sqrt{3}\,\hat{\sigma}$$

(b) $L(\hat{a},\hat{b}) = \left(\frac{1}{\hat{b}-\hat{a}}\right)^n, \quad \hat{a} \le X_1,\dots,X_n \le \hat{b}$

$\begin{cases} \frac{\partial \ell}{\partial \hat{a}} = 0 \\ \frac{\partial \ell}{\partial \hat{b}} = 0 \end{cases}$: has no solution, but to maximize $\frac{1}{(\hat{b}-\hat{a})^n}$ directly, you need to minimize $\hat{b}-\hat{a}$,

so $\begin{cases} \hat{b} = \max\{X_{1..n}\} \text{ is the smallest} \\ \hat{a} = \min\{X_1,\dots,X_n\} \text{ is the largest possible} \end{cases}$

(c) $\tau = \mathbb{E}(X_i) = \frac{a+b}{2} \Rightarrow \hat{\tau} = \frac{\hat{a}+\hat{b}}{2}$ by the equivariance principle.

(A.)

(a) $\ell(p) = \sum_{i=1}^{n} \left[(X_i - 1)\ln(1-p) + \ln p\right] = \left(\left(\sum X_i\right) - n\right)\ln(1-p) + n\ln p$

$$\ell'(p) = -\frac{(\sum X_i)-n}{1-p} + \frac{n}{p} = 0 \Rightarrow \frac{(\sum X_i)-n}{1-p} = \frac{n}{p} \Rightarrow$$

$$\Rightarrow p\left[\sum (X_i) - n\right] = (1-p)\,n \Rightarrow \boxed{\hat{p} = \frac{n}{\sum X_i} = \frac{1}{\bar{X}}}$$

(b) $\frac{\partial \ln f}{\partial p} = -\frac{X-1}{1-p} + \frac{1}{p}, \quad \frac{\partial^2 \ln f}{\partial p^2} = -\frac{(X-1)}{(1-p)^2} - \frac{1}{p^2},$

$$I(\underset{\substack{\uparrow \\ p}}{\theta}) = \frac{\mathbb{E}(X)-1}{(1-p)^2} + \frac{1}{p^2} = \frac{\frac{1}{p}-1}{(1-p)^2} + \frac{1}{p^2} = \frac{p-p^2+(1-p)^2}{p^2(1-p)^2} =$$

$$= \frac{1}{p^2(1-p)}$$

B. (a) $\cancel{\partial \ell}$ $\ell(\beta, c) = \sum_{i=1}^{n} \left[-\ln\beta - \frac{X_i - c}{\beta} \right]$

$$= -n\ln\beta + \frac{nc}{\beta} - \frac{\sum X_i}{\beta} \quad (\forall i\ X_i \geq c)$$

$$\begin{cases} \frac{\partial \ell}{\partial \beta} = -\frac{n}{\beta} - \frac{nc}{\beta^2} + \frac{\sum X_i}{\beta^2} = 0 \Rightarrow \boxed{\hat{\beta} = \frac{\sum X_j}{n} - \hat{c}} \\ \frac{\partial \ell}{\partial c} = \frac{n}{\beta} \overset{?}{=} 0 \end{cases}$$

has no solution, but maximizing $\frac{nc}{\beta}$ directly, we obtain $\hat{c} = \min(X_i)$

$$\Rightarrow \hat{\beta} = \bar{X} - \min(X_i)$$

(b) Does not exist because $\frac{\partial \ell}{\partial c}$ does not exist at $c = X_i$.

C. (a) $\hat{\sigma} = \sqrt{\frac{\sum (X_i - \theta)^2}{n}}$ (done in class)

(b) $\ell(\psi) = \sum_{i=1}^{n} \left(-\psi - \frac{X_i^2}{2e^{2\psi}} \right) = -n\psi - \frac{\sum X_i^2}{2e^{2\psi}}$

+ const

$$\ell'(\psi) = -n + \left(\sum X_i^2\right) e^{-2\psi} = 0 \Rightarrow e^{-2\psi} = \frac{n}{\sum X_i^2}$$

$$\Rightarrow \hat{\psi} = \frac{1}{2}\ln\left(\frac{\sum X_i^2}{n}\right) = \ln\hat{\sigma} \quad \checkmark$$