Hw 3 key



4, 8, 3

(3.) $P(\max(X_1,\dots,X_n) \le y) = y^n = F_Y(y)$, $0 \le y \le 1$

$f(y) = F'(y) = ny^{n-1}$, $\mathbb{E}(Y) = \int_0^1 y \cdot ny^{n-1}\,dy = \frac{n}{n+1}$

(10.) $$\mathbb{E}(e^X) = \int_{-\infty}^{\infty} e^x \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-1)^2}{2}} e^{\frac{1}{2}}\,dx$$

$$= \sqrt{e} \boxed{\int_{-\infty}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\,dy} = \sqrt{e}$$

$\boxed{y = x-1 \quad dy = dx}$

$= 1$ because $\frac{1}{\sqrt{2\pi}} e^{-y^2/2}$ is a $N(0,1)$ density

$$\mathbb{E}((e^X)^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{+2x} e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-2)^2}{2}} e^2\,dx$$

$$= e^2;\quad Var(Y) = E(Y^2) - (\mathbb{E}Y)^2 = e^2 - (\sqrt{e})^2 = e(e-1)$$

(12.) See Math 382 Book

a) Uniform: $\mathbb{E}(X) = \int_a^b \frac{x\,dx}{b-a} = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$

$\mathbb{E}(X^2) = \int_a^b \frac{x^2\,dx}{b-a} = \frac{b^3-a^3}{3(b-a)} = \frac{a^2+ab+b^2}{3}$, $V(X) = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4}$

$$= \frac{4a^2+4ab+4b^2-3a^2-3b^2-6ab}{12}$$

$$= (b-a)^2/12$$

b) Gamma

$$\mathbb{E}(X) = \int_0^{\infty} \frac{x \cdot x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha}\,dx$$

$$= \beta \int_0^{\infty} \frac{y^\alpha e^{-y}\,dy}{\Gamma(\alpha)} = \beta\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \alpha\beta$$

$\boxed{y = \frac{x}{\beta} \quad dy = \frac{dx}{\beta}}$

$$E(X^2) = \int_0^{\infty} \frac{x^{\alpha+1} e^{-x/\beta}\,dx}{\Gamma(\alpha)\beta^\alpha}$$

$$= \beta^2 \int_0^{\infty} \frac{y^{\alpha+1}e^{-y}\,dy}{\Gamma(\alpha)} = \beta^2\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}$$

$$= \beta^2(\alpha+1)(\alpha),\quad V(X) = \mathbb{E}(X^2) - \mu^2 = \beta^2\alpha^2 + \beta^2\alpha - (\alpha\beta)^2$$

$$= \alpha\beta^2$$

(23.) $$\mathbb{E}(e^{tX}) = \sum_{k=0}^{\infty}\left(e^{tk}\,\frac{e^{-\mu}\mu^k}{k!}\right) = e^{-\mu}\sum_{k=0}^{\infty}\frac{(e^t\mu)^k}{k!}$$

$$= e^{-\mu}e^{\mu e^t} = e^{\mu(e^t-1)}$$ Poisson

$$\mathbb{E}(e^{tX}) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

$$\underset{\substack{\uparrow \\ z=\left(\frac{x-\mu}{\sigma}\right) \\ dz=\frac{dx}{\sigma}}}{=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{t(\mu+\sigma z)}\,e^{-z^2/2}\,dz$$

$$= e^{t\mu}\cancel{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(z-\sigma t)^2}{2}}\,e^{\frac{\sigma^2t^2}{2}}\,dz$$

$$= e^{t\mu}e^{\sigma^2t^2/2}$$ Normal

(A.)

a) $V(X+2Y) = V(X) + 4V(Y) + 4\,\mathrm{Cov}(X,Y)$

$= 5 + 4(5) + 4(0.8\sqrt{5}\sqrt{5}) = 41$

b) $\mathrm{Cov}(X-2Y, X+2Y) = V(X) - 4V(Y) = -15$