Exam 2 Name (please print) KEY

Math 483, Fall 2017 November 21, 2017

Show work and correct notation for full credit. Give numerical or simple fraction answers whenever possible.

Problem	1	2	3	4	5	6	Total
Earned							
Possible	10	10	10	10	10	10	50

Solve 5 problems out of 6. Cross one out.

1. The success percentages in two groups are p_1 and p_2 respectively. Two groups of 80 observations each were collected. The first group had 37 successes, and second group had 24 successes.

(a) Find a 95% confidence interval for $p_1 - p_2$, based on Normal approximation.

(b) Based on the above CI, do you believe that $p_1 = p_2$?

(c) Perform the Wald test for $H_0 : p_1 = p_2$. Also, find the p-value.

(a) $\hat{p}_1 - \hat{p}_2 \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$ is 0.1625 ± 0.1484 or $\approx [.01, .31]$

(b) No, the C.I. does not contain 0.

(c) $W = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}} \simeq 2.15$

p-value $= 2P(Z > 2.15) \simeq .032$

Reject H_0: p_1 is higher than p_2!

2. (a) Let $\hat{\theta}$ be asymptotically Normal(mean $=17.1$, variance $= 4.5/n$). Use Delta method to find the approximate distribution of $\ln(\hat{\theta})$

(b) Give an example of an estimation problem where MLE is not asymptotically Normal.

(a) $\approx N\left(\ln(17.1),\ \text{var} = \left(\frac{1}{\theta}\right)^2 \frac{4.5}{n}\right)$

$g(\theta) = \ln(\theta)$
$g'(\theta) = \frac{1}{\theta}$

$\approx N\left(2.84, \frac{0.015}{n}\right)$

(b) $X_1, \dots, X_n \sim$ Uniform$[0, \theta]$ is not a regular estimation problem. $\hat{\theta}$ (MLE) is $\max\{X_1, \dots, X_n\}$, $P(\hat{\theta} < x) = \left(\frac{x}{\theta}\right)^n$, does not converge to Normal dist.

3. MLE
Let the i.i.d. sample $X_1, \dots, X_n$ come from the Poisson distribution with unknown mean $\theta > 0$.

(a) Find the MLE for θ. Evaluate numerically based on a sample of size $n = 100$, and $\sum_{i=1}^{100} X_i = 815$.

(b) Find the Fisher information

(c) Describe the asymptotic distribution of the MLE as n gets large.

(a) $\mathcal{L}(\theta) = \prod_{i=1}^{n} \frac{e^{-\theta}\theta^{X_i}}{(X_i)!}$

$\ell(\theta) = \text{const} + \sum[-\theta + X_i \ln\theta]$

$\ell'(\theta) = -n + \frac{\sum X_i}{\theta} = 0 \Rightarrow \hat{\theta} = \frac{\sum X_i}{n} = \bar{X} = 8.15$

(b) $\ln f(\theta) = -\theta + X\ln\theta + \text{const}$,

$\frac{\partial \ln f}{\partial\theta} = -1 + \frac{X}{\theta}$, $\frac{\partial^2 \ln f}{\partial\theta^2} = -\frac{X}{\theta^2}$

$I(\theta) = -E\left(\frac{\partial^2 \ln f}{\partial\theta^2}\right) = \frac{EX}{\theta^2} = \frac{\theta}{\theta^2} = \frac{1}{\theta}$, $I_n(\theta) = \frac{n}{\theta}$

(c) $\hat{\theta} \approx N\left(\theta, \text{var} = \frac{\theta}{n}\right)$

$\mathcal{L}(\theta) = \prod \frac{1}{\theta} e^{-X_i/\theta} \Rightarrow \ell(\theta) = \sum\left(-\ln\theta - \frac{X_i}{\theta}\right)$; recall that MLE $= \overline{X}$

$\lambda = 2\left[\ell(\hat\theta) - \ell(\theta_0)\right] = -2n\left[\ln \bar{X} + \frac{\bar X}{\bar X} - \ln\theta_0 - \frac{\bar X}{\theta_0}\right]$ $\quad -n\left[\ln\theta + \frac{\bar X}{\theta}\right]$

4. For the Exponential distribution with unknown mean $\theta = \beta$ perform the likelihood ratio test for

$$\begin{cases} H_0: & \theta = 10 \\ H_1: & \theta \neq 10 \end{cases}$$

$= 6.35$. At $\alpha = 0.05$

based on the data: $n = 50$, $\overline{X} = 14$.

$\chi^2_{1,.95} = 3.841$

$6.35 > 3.84 \Rightarrow$ Reject H_0

5. A lottery can have 3 kinds of prizes, A, B and C. We'd like to test the hypothesis that Prize B and Prize C are equally likely, while Prize A is twice as likely as each of B and C. The empirical data on prizes gave the following counts:

A	B	C
38	12	16

$n = 66$

E_i:	A	B	C
	33	16.5	16.5

$\chi^2 = \frac{(38-33)^2}{33} + \dots + \frac{(16-16.5)^2}{16.5} = 2.0$

Perform the chi-square test. Give your conclusion at the level $\alpha = 0.1$.

Table C: Critical points of the chi-square distribution

	Upper tail probability						
Degrees of freedom	0.100	0.050	0.025	0.010	0.005	0.001	0.0005
1	2.706	3.841	5.024	6.635	7.879	10.828	12.116
2	4.605	5.991	7.378	9.210	10.597	13.816	15.202
3	6.251	7.815	9.348	11.345	12.838	16.266	17.730
4	7.779	9.488	11.143	13.277	14.860	18.467	19.997
5	9.236	11.070	12.833	15.086	16.750	20.515	22.105

$\chi^2_{.10,\, df=2} = \cancel{5.991}\ 4.605$

$\chi^2 < \cancel{5.991}\ 4.605$

$\Rightarrow$ Accept H_0

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6.

[3] a) i. ii. iii.

median: $F(m) = 1 - e^{-m/\beta} = \frac{1}{2}$

[3] b) $\hat\beta = 2.35$, $\Rightarrow m = \ln 2\, \beta$

$\Rightarrow \hat m = \ln 2\,(2.35) \simeq 1.63$

[2+2] c) i. False (Inverse of II needed)

ii. True (equivariance principle)