Exam 2 Name (please print) KEY

Math 483, Fall 2019 November 21, 2019

Show work and correct notation for full credit. Give numerical or simple fraction answers whenever possible.

Problem	1	2	3	4	5	6	Total
Earned							
Possible	10	10	10	10	10	10	50

① a) $\hat{\mu}_1 = \bar{X}$, $\hat{\mu}_2 = \bar{Y}$
$\hat{se}(\hat{\mu}_1) = \frac{\hat{\sigma}}{\sqrt{n}} = \frac{\bar{X}}{\sqrt{n}}$, $\hat{se}(\hat{\mu}_2) = \frac{\bar{Y}}{\sqrt{n}}$

b) $\bar{X} - \bar{Y} \pm z_{\alpha/2}\sqrt{\frac{(\bar{X})^2}{n} + \frac{(\bar{Y})^2}{n}}$
$= 0.6 \pm 1.96\sqrt{\frac{5.5^2}{100} + \frac{4.9^2}{100}}$
$= 0.6 \pm 1.44$

Solve 5 problems out of 6. Cross one out.

(c) yes, $\mu_1 - \mu_2 = 0$ is in the C.I.

(d) $W = \frac{0.6}{\sqrt{\frac{5.5^2}{100} + \frac{4.9^2}{100}}} = 0.814$
$|W| < z_{\alpha/2} = 1.96$
$\Rightarrow$ accept H_0 at $\alpha = .05$; p-value $= 2P(\cancel{Z} > .814) = 2(.5 - .2910) \approx .42$

1. Suppose $X_1, ..., X_n \sim$ `Exponential`(μ_1), $Y_1, ..., Y_n \sim$ `Exponential`(μ_2) and $n = 100$. The sample observations yielded $\overline{X} = 5.5$ and $\overline{Y} = 4.9$.

(a) Find point estimates $\hat{\mu}_i$ and $\widehat{\mathtt{se}}(\hat{\mu}_i)$, $i = 1, 2$.

(b) Find a 95% confidence interval for $\mu_1 - \mu_2$, based on Normal approximation.

(c) Based on the above CI, do you believe that $\mu_1 = \mu_2$?

(d) Perform the Wald test for $H_0 : \mu_1 = \mu_2$.

2. MLE

Let X have the Binomial distribution with given n and unknown $p = \theta$.

(a) Find the MLE for θ. [Show the derivation!]

(b) Find the Fisher information [Show the complete derivation!].

(c) Describe the asymptotic distribution of the MLE as n gets large.

② a)
$\mathcal{L}(\theta) = \binom{n}{X}\theta^X(1-\theta)^{n-X}$
$\ell(\theta) = \text{const} + X\ln\theta + (n-X)\ln(1-\theta)$
$\ell'(\theta) = \frac{X}{\theta} - \frac{n-X}{1-\theta} = 0$
$\Rightarrow X(1-\theta) = \theta(n-X)$
$\Rightarrow \hat{\theta} = \frac{X}{n} = \hat{p}$

b) $I_n(\theta) = -\mathbb{E}(\ell''(\theta)) = -\mathbb{E}\left(-\frac{X}{\theta^2} - \frac{n-X}{(1-\theta)^2}\right) = \frac{n\theta}{\theta^2} + \frac{n-n\theta}{(1-\theta)^2}$
$= n\left(\frac{1}{\theta} + \frac{1}{1-\theta}\right) = \frac{n}{\theta(1-\theta)}$

c) $\hat{\theta} \approx \mathcal{N}\left(\theta, \frac{1}{I_n(\theta)}\right) = \mathcal{N}\left(\theta, \frac{\theta(1-\theta)}{n}\right)$

(3.) $\lambda = 2\ln\left(\frac{L(\hat\theta)}{L(\theta_0)}\right)$, recall $\hat\theta = \bar X$ for Normal.

$\ln L = \text{const} - \frac{\sum(X_i-\theta)^2}{2} \Rightarrow \lambda = \sum(X_i-0)^2 - \sum(X_i-\bar X)^2 = \sum X_i^2 - \sum X_i^2 + 2\bar X(\sum X_i) - n(\bar X)^2 = n(\bar X)^2 = (\sum X_i)^2/n = 3.92,$

Use $\chi^2(1)$ table, at $\alpha = .05$ $\chi^2_{1,\alpha} = 3.841$

$3.92 > 3.84$ $\Rightarrow$ Reject H_0 at $\alpha = 5\%$

3. For the Normal distribution with unknown mean $\mu = \theta$ and known $\sigma = 1$, perform the likelihood ratio test for

$$\begin{cases} H_0: & \mu = 0 \\ H_1: & \mu \neq 0 \end{cases}$$

based on the data: $n = 50$, $\sum X_i = 14$, $\sum X_i^2 = 240$.

Extra credit

E.C.

$\beta(\theta) = 1 - \Phi\left(\frac{\theta_0-\theta_1}{se} + z_{\alpha/2}\right) + \Phi\left(\frac{\theta_0-\theta}{se} - z_{\alpha/2}\right) = 1 - \Phi(1.41+1.645) + \Phi(1.41-1.645) \approx 0.41$

Calculate the power of the Wald test for the above, with $n = 50$, $\mu = -0.2$ and $\alpha = 0.1$

4. In a genetic study, the subjects were classified into 4 groups, with the counts as follows

Group	1	2	3	4	Total
Count	22	20	30	28	100
E_i	25	25	25	25	

Can the groups be equally distributed among the population? Test at the level $\alpha = 0.05$.

(4.) $H_0: p_1 = p_2 = p_3 = p_4 = 1/4$

$\chi^2 = \sum \frac{(X_i - E_i)^2}{E_i} = \frac{(22-25)^2}{25} + \dots + \frac{(28-25)^2}{25} = 2.72$

$Df = k - 1 = 3$

Table C: Critical points of the chi-square distribution

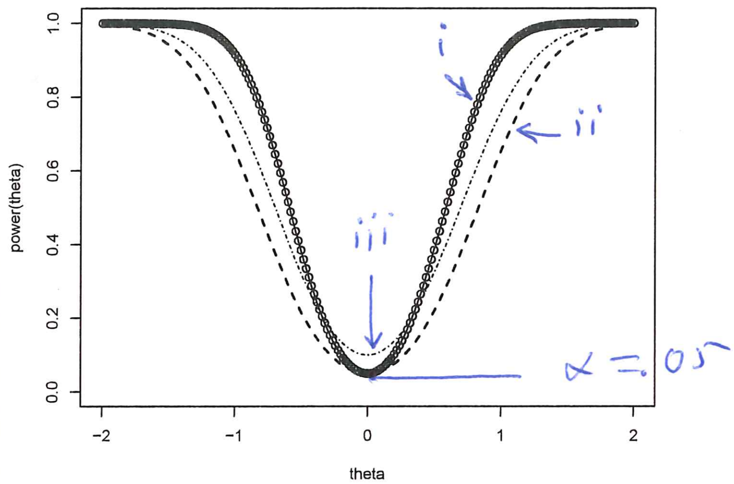
Degrees of freedom	Upper tail probability 0.100	0.050	0.025	0.010	0.005	0.001	0.0005
1	2.706	3.841	5.024	6.635	7.879	10.828	12.116
2	4.605	5.991	7.378	9.210	10.597	13.816	15.202
3	6.251	7.815	9.348	11.345	12.838	16.266	17.730
4	7.779	9.488	11.143	13.277	14.860	18.467	19.997
5	9.236	11.070	12.833	15.086	16.750	20.515	22.105

at $\alpha = .05$, $\chi^2_{.05,3} = 7.815 > 2.72$

$\Rightarrow$ Accept H_0, no evidence of difference.

5. (a) Match each of the power curves below with one of the following cases:

i. $n = 100, \alpha = 0.05$ ii. $n = 50, \alpha = 0.05$ iii. $n = 50, \alpha = 0.1$



(b) $g(\theta_1, \theta_2) = \theta_1 \theta_2$

$\hat{\theta}_1 \hat{\theta}_2 \approx N(\theta_1 \theta_2, (\nabla g)^T J (\nabla g))$

$\nabla g = \begin{bmatrix} \theta_2 \\ \theta_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$J = I^{-1} = \frac{1}{n} \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$

$\Downarrow$

$\hat{\theta}_1 \hat{\theta}_2 \approx N\left(3, \frac{13}{2n}\right)$

(b) $\hat{\boldsymbol{\theta}} = [\hat{\theta}_1, \hat{\theta}_2]^T$ is an MLE for some estimation problem with the Fisher information matrix

$$\mathbf{I} = n \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$$

Assuming that the "true values" are $\theta_1 = 1$ and $\theta_2 = 3$, use Delta method to find the approximate distribution of $\hat{\theta}_1 \hat{\theta}_2$

6. Misc.

(4 pt)

(a) Give an example of an estimation problem for which MLE is not unbiased.

$X_i \sim N(\mu, \sigma^2)$
$\hat{\sigma}^2 = \frac{\sum (X_i - \bar{X})^2}{n}$ is biased

OR

$\hat{\theta}$ for $\text{Uniform}[0, \theta]$
$\hat{\theta} = \max\{X_1, \ldots, X_n\}$
$\mathbb{E}(\hat{\theta}) < \theta$

True or False? Explanations are not necessary, but they won't hurt.

2 pt each

(b) Fisher information matrix can be used to make a quadratic approximation to the log-likelihood surface. **T**

(c) Since $\hat{\sigma}^2$ is an MLE for σ^2, then $\hat{\sigma}$ is a consistent estimate for σ. **T** (MLE is consistent)

(d) The problem of finding the MLE for θ when $X_1, \ldots, X_n \sim \texttt{Uniform}[0, \theta]$ is an example of a regular estimation problem.

F (Fisher info. DNE, $\hat{\theta}$ not asymp. Normal)