Midterm Exam   Name (please print) ____________

Math 483, Fall 2019   October 10, 2019

Show work and correct notation for full credit. Give numerical or simple fraction answers whenever possible.

(1.) a) Yes, $X_n \xrightarrow{P} 1$ because $X_n - 1 \sim$ Uniform $[-1/n, 1/n]$

$P(|X_n - 1| > \epsilon) = 0$ for large enough n.

Problem	1	2	3	4	5	6	Total
Earned							
Possible	10	10	10	10	10	10	50

b) many answers; for example, $X_n \sim$ Uniform $[\frac{1}{n}, 1-\frac{1}{n}]$ $\xrightarrow{D}$ Uniform $[0,1]$

Solve 5 problems out of 6. Cross one out.

1. (a) Does the sequence $X_n \sim \texttt{Uniform}[a = 1-1/n, b = 1+1/n]$ converge in probability? If no, explain why not. If yes, find the limit and explain why the sequence converges to it.

(b) Give an example of something converging in distribution to a non-Normal distribution. Explain.

(2.) $29.5 \pm 1.88 \frac{29.5}{\sqrt{180}} = 29.5 \pm 4.1$, or $[25.4, 33.6]$

$z_{\alpha/2} = 1.88$ for $\alpha/2 = .03$ $(TA(z_{\alpha/2}) = .47)$, $\hat{\sigma} = \hat{\theta} = 29.5$

2. Let $X_1, ..., X_n$ be an i.i.d. sample from Exponential distribution with parameter (mean) θ, with $n = 180$ and $\overline{X} = 29.5$.
Find a 94% C.I. for θ.

(3.) a) $V(X_1) + V(X_2) = 20$ b) $V(X_1) + V(X_2) + 2\,Cov(X_1, X_2) = 10 + 10 + 2 \cdot \sigma_1 \sigma_2 \rho = 20 - 10 = 10$

3. Let $X_i \sim \mathcal{N}(0, var = 10)$. Find $Var(X_1 + X_2)$ assuming:

(a) X_i are independent

(b) $corr(X_1, X_2) = -0.5$

(4.) a) $P(29.5 < \overline{X} < 30.5) \simeq P\left(\frac{29.5-30}{\sqrt{6/90}} < Z < \frac{30.5-30}{\sqrt{6/90}}\right) = P(-1.94 < Z < 1.94) = 2 \cdot .474 \simeq .948$

4. Let $\overline{X}$ be a mean of an i.i.d. sample $X_1, ..., X_n$, $n = 90$, with $\mathbb{E}(X_i) = 30$ and $Var(X_i) = 6$.

b) $\overline{X} \approx \mathcal{N}(30, \frac{6}{90})$

(a) Approximate the probability that $\overline{X}$ is between 29.5 and 30.5

$g(x) = \ln x$, $g'(x) = \frac{1}{x}$

(b) Use Delta-method to find the approximate distribution for $\ln(\overline{X})$.

$g(\overline{x}) \approx \mathcal{N}\left(g(30), [g'(30)]^2 \frac{6}{90}\right) = \mathcal{N}(3.4, 7.4 \times 10^{-5})$ or $\sigma \approx .0086$

(a) $\hat{p} - \hat{q} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n} + \frac{\hat{q}(1-\hat{q})}{m}}$

$\hat{p} = 48/85 \simeq 0.565$

$\hat{q} = 26/75 \simeq 0.347$

$z_{\alpha/2} = 2.576$

5. (a) We are testing the efficacy of a new sleep medicine. After taking the medicine, 48 out of 85 subjects reported improved sleep. After taking the placebo (a pill that doesn't really do anything), 26 out of 75 subjects reported improved sleep. Find a 99% confidence interval for **the difference** in proportions, p_1 = proportion of people helped by the medicine, and p_2 = proportion of people helped by the placebo.

$\Rightarrow 0.218 \pm 2.576\sqrt{.0059}$, 0.218 ± 0.198 or $[.020, .416]$

(b) The *kurtosis* is defined as

$$\gamma = \frac{\mathbb{E}\left[(X - \mathbb{E}X)^4\right]}{[Var(X)]^2}$$

It measures how heavy are the distribution's "tails". Normal distribution, for example, has $\gamma = 3$.
Show that for any a and b, $\gamma(a + bX) = \gamma(X)$.

(b) $E\left[\{a+bX - E(a+bX)\}^4\right]$
$= E\left[\{bX - bE(X)\}^4\right]$
$= b^4 E\left[(X - EX)^4\right]$,
$Var(a+bX) = b^2 Var(X)$

$\Rightarrow \gamma(a+bX) = \frac{b^4 E[(X-EX)^4]}{b^4 [Var(X)]^2} = \gamma(X)$

6. True or False? Explanations are not necessary, but they won't hurt.

(a) $S^2 = \frac{\cdots}{n-1}$ is unbiased

F (a) $\tilde{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n+1}$ is an unbiased estimate of the variance σ^2.

(b) yes, $\hat{F}_n(x)$ is like $\hat{p}$, $\approx N\left(\hat{p}, \frac{\hat{p}(1-\hat{p})}{n}\right)$

T (b) Let $\hat{F}_n(x)$ be the empirical CDF for a sample $X_1, ..., X_n$, with theoretical CDF $F(x)$. Then, for any x such that $0 < F(x) < 1$, $\hat{F}_n(x)$ is an asymptotically normal estimate of $F(x)$.

(c) common knowledge

T (c) If two random variables X_1 and X_2 are independent, then the covariance between them is 0.

(d) yes (Theorem 5.5 (b)) $\sqrt{x}$ is a continuous function

(d) If $\hat{\theta}_n$ is a consistent estimate of θ, then $\sqrt{\hat{\theta}_n}$ is a consistent estimate of $\sqrt{\theta}$ (assume everything is positive). T

(e) In the CLT, if the "true" variance σ^2 is replaced by sample variance S^2, then the result still converges to standard Normal, that is,

$$\frac{\overline{X} - \mu}{S/\sqrt{n}} \xrightarrow{D} N(0, 1)$$

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(e) Theorem 5.5 (e)

$\left(\frac{\sigma}{S} \xrightarrow{P} 1\right)$