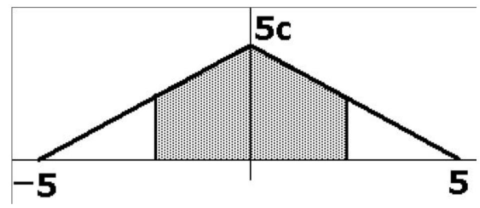
Math 382
Practice Exam 2 KEY

Consider the PDF of a random variable X



1.
$$f(x) = \begin{cases} c(5+x) & -5 \le x \le 0 \\ c(5-x) & 0 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find c that makes $f(x)$ a legitimate density function. Plot $f(x)$.
Total area under the graph is 25c, therefore c = 1/25

(b) Find the probability that X is between -2.5 and 2.5
Shaded area is $1 - 2 *$ (area of triangle) $= 1 - 2(2.5 * 2.5c)/2 = 0.75$

(c) Find $\mathbb{E}(X)$
Since the graph is symmetrical about 0, the mean is 0

(d) Find $V(X)$

$$V(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{2}{25}\int_0^5 x^2(5-x)\,dx = \frac{2}{25}\left[\frac{5}{3}x^3 - \frac{1}{4}x^4\right]_0^5 = 25/6$$

(e) Compute the CDF of X.
For $x < 0$:

$$F(x) = \frac{1}{25}\int_{-5}^{x}(5+t)\,dt = \frac{1}{25}\left[5t + \frac{t^2}{2}\right]_{-5}^{x} = \frac{1}{2} + x/5 + x^2/50$$

check that $F(-5) = 0$ and $F(0) = 0.5$

For $0 < x < 5$:
$$F(x) = 1/2 + x/5 - x^2/50 \qquad \text{(Why?)}$$

2. In the past few years, the average daily temperature in Socorro on October 10 was 58.5^oF with standard deviation 9.5^oF. Assume Normal distribution.

(a) What are the chances that on an October 10, the average daily temperature will be above 40^oF?
$$P(X > 40) = P\left(Z > \frac{40 - 58.5}{9.5}\right) = 0.974$$

(b) Find the interval $[a, b]$ symmetric around the mean, such that $P(a \le X \le b) = 0.98$
Find z^* such that $TA(z^*) = 0.98/2 + 0.5 = 0.99$, so that $z^* = 2.33$.
Then, $a = \mu - \sigma z^* = 36.4$ and $b = \mu + \sigma z^* = 80.6$.

3. The number of computer crashes at a Speare lab is believed to have Poisson distribution with the intensity of 0.2 crashes per hour.

(a) Find the distribution of times between consecutive crashes.
Exponential with mean = 1/intensity = 5

(b) Find the probability that the lab will go 7 hours without a crash.
$P(X > 7) = exp(-7/5) = 0.2466$

(c) Find the probability that the first crash will occur between 2 and 8 hours.
$P(2 < X < 8) = exp(-2/5) - exp(-8/5) = 0.4684$

(d) Describe the distribution of the time when 3rd crash occurs.
Gamma $(\alpha = 3, \beta = 5)$ (sum of 3 exponentials, each with mean 5)

4. Using Normal approximation (even without continuity correction ± 0.5). First, $\mu = 0.15(200) = 30$ and $\sigma = \sqrt{200 * 0.15(1 - 0.15)} = 5.05$ Thus,

$$P(X \geq 25) \approx P\left(Z > \frac{25 - 30}{5.05}\right) = 1 - TA(-0.99) = 0.84$$

5. The following is a table of joint distribution of X and Y

Y	0	1	
$X = 0$	0.25	0.35	**0.6**
$X = 1$	0.1	0.3	**0.4**
	0.35	**0.65**	

(a) Are X, Y independent? Explain.
They are not: for example, $P(X = 0, Y = 0) = 0.25$, but $P(X = 0) \cdot P(Y = 0) = 0.6 * 0.35 = 0.21$

(b) Fill in the marginal distributions of X, Y. **Done**

(c) Find $\mathbb{E}(X)$ and $\mathbb{E}(Y)$

$$\mathbb{E}(X) = 0 * 0.6 + 1 * 0.4 = 0.4 \qquad \mathbb{E}(Y) = 0.65$$

(d) Find $Cov(X, Y)$

$$\mathbb{E}(XY) = 0.3, \qquad Cov(X, Y) = 0.3 - 0.4 * 0.65 = \mathbf{0.04}$$

these are done for X=0 and X=1, however covariance should be the same

6. A ... ance is
e ... andard
d

see Part 2

... er).

... $, R_j) =$

#7. see also Part 2

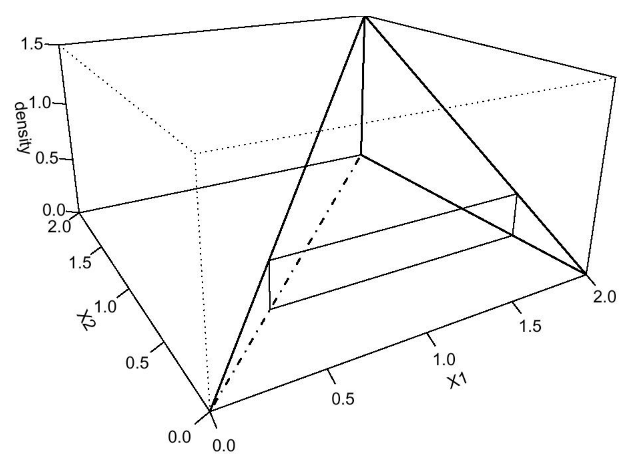
7. Suppose that X and Y have joint density $f(x, y) = ky$, for $0 \le y \le x \le 2$

(a) find the constant k that makes f a legitimate density function.

$$1 = \int_0^2 \int_y^2 f(x, y)\, dx\, dy$$

is the volume of pyramid with its base a triangle in (X, Y) plane, height $h = 2k$.

$$1 = Volume = \frac{Ah}{3} = \frac{4}{3}k, \qquad \text{thus } k = 3/4$$

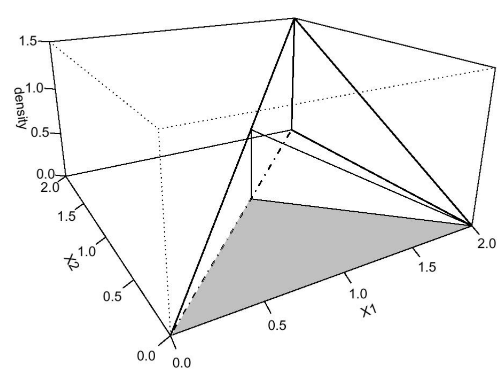


(b) find $P(X + Y < 2)$

See small plot. Volume of pyramid over shaded triangular region $= 1/4$

(c) find the conditional density of X given that Y equals 0.5

See large plot. When $y = 0.5$, cross-section gives the rectangle $0.5 \le x \le 2$.
Thus, X is uniform on $0.5 \le x \le 2$, and its density $f(x) = 1/(2 - 0.5) = 2/3$.



(d) Find the marginal density of X. Are X and Y independent?

$$f_X(x) = \int_0^x 0.75y\, dy = 3x^2/8, \qquad 0 \le x \le 2$$

Clearly X and Y are not independent because $Y \le X$.

(e)

$$\int_0^2 \int_y^2 x \sin(y) * 0.75y\, dx\, dy$$

8. The life length of fuses is Exponential with mean = 300 hours. Assume that fuses behave independently.

(a) Find the joint density for lifetimes of two such fuses.

By independence,

$$f(x_1, x_2) = f_1(x_1) f_2(x_2) = \left(\frac{1}{300} e^{-x_1/300}\right) \left(\frac{1}{300} e^{-x_2/300}\right)$$

(b) Find $P(\text{Total life of 2 such fuses}) \leq 500$ hours.
Let's work in 100 hour units.

$$P(X_1 + X_2 \leq 5) = \int_0^5 \left[\int_0^{5-x_2} \frac{1}{3} e^{-x_1/3}\, dx_1\right] \frac{1}{3} e^{-x_2/3}\, dx_2 =$$

$$= \int_0^5 \frac{1}{3} e^{-x_2/3} \left(1 - e^{-5/3} \cdot e^{x_2/3}\right) dx_2 = \int_0^5 \frac{1}{3}(e^{-x_2/3} - e^{-5/3})\, dx_2 = 1 - e^{-5/3} - \frac{5}{3} e^{-5/3} = 0.496$$

We could also use Poisson process, since $X_1 + X_2$ is $Gamma(\alpha = 2, \beta = 300)$. Let N be number of fuse burnouts on $[0, 500]$ interval. It is Poisson with the mean $\mu = \lambda t = 5/3$.

$$P(X_1 + X_2 < 500) = P(N \geq 2) = 1 - f(0) - f(1) = 1 - e^{-5/3} - \frac{5}{3} e^{-5/3}$$