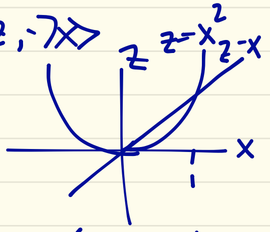
23. $\vec{F} = \langle 7xy, -z, 3xyz \rangle$

$x=y,\ x=z,\ x^2=z$ $\qquad$ $n=\langle 1,-1,0\rangle$

$$\text{curl}F = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 7xy & -z & 3xyz \end{vmatrix} = \langle 3xz+1, -(3xz+0), 0-7x \rangle$$

$$= \langle 3xz+1, -3xz, -7x \rangle$$



parameterize

$\langle 3xz+1, -3xz, -7x \rangle \cdot \langle 1,-1,0 \rangle = 6xz+1$

$$\int_0^1 \int_{x^2}^{x} (6xz+1)\,dz\,dx$$

$$\iint \vec{F} \cdot \vec{n}\,dS$$

33. $\left\langle \dfrac{x}{x^2+y^2}, \dfrac{-y}{x^2+y^2}, z^2 \right\rangle = F$ $\qquad$ C is unit circle

- cont. diff. on open set containing S
- C be simple, smooth closed
- S be oriented and smooth

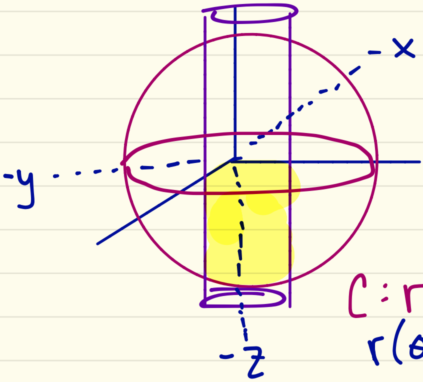
$C: x^2+y^2=1$ curve in the plane $z=0$

$x^2+y^2 \neq 0$

vector field F isnt defined on z-axis. But any S of which c is a boundary must include at least one pt. on z

25. $\iint_S \operatorname{curl} F \cdot n \, dS$ S is the "cap" of the unit sphere below the xy-plane and inside $x^2+y^2=1/9$ (cylinder) with the outward normal

$$F(x,y,z) = -yz^2 i + xz^2 j + 3^{-xyz} k$$



$\int \vec{F} \cdot d\vec{r}$ $\quad 0 \le \theta \le 2\pi$

$x^2+y^2+z^2=1$, $x^2+y^2=1/9$

$z = -\frac{2\sqrt{2}}{3}$

keep surface on (L)

$C: r(\theta) = \langle \frac{1}{3}\cos\theta, \frac{1}{3}\sin(-\theta), -\frac{2\sqrt{2}}{3} \rangle$

$r(\theta) = \langle \frac{1}{3}\cos\theta, -\frac{1}{3}\sin\theta, -\frac{2\sqrt{2}}{3} \rangle$

Upside-down ↳ CW

$r(\theta) = \langle \frac{1}{3}\sin\theta, \frac{1}{3}\cos\theta, -\frac{2}{3}\sqrt{2} \rangle$

$F = \langle -\frac{1}{3}\cos\theta\left(-\frac{2}{3}\sqrt{2}\right)^2, \frac{1}{3}\sin\theta \cdot \left(-\frac{2}{3}\sqrt{2}\right)^2, 3^{-\frac{1}{3}\sin\theta \frac{1}{3}\cos\theta\left(-\frac{2}{3}\sqrt{2}\right)}$

$d\vec{r} = \langle \frac{1}{3}\cos\theta, -\frac{1}{3}\sin\theta, 0 \rangle$

$\bar{F} \cdot d\bar{r} = -\frac{1}{9}\cos^2\theta \cdot \frac{8}{9} - \frac{1}{9}\sin^2\theta \cdot \frac{8}{9} + 0$

$= -\frac{8}{81}(1)$

$$\int_0^{2\pi} -\frac{8}{81} \, d\theta = -\frac{8}{81}(2\pi) = -\frac{16\pi}{81}$$

Stoke's $\to$ do opposite

1. Find *curl* $\vec{F}$ and *div* $\vec{F}$ if $\vec{F}(x,y,z) = e^{-x}\sin y\,\mathbf{i} + e^{-y}\sin z\,\mathbf{j} + e^{-z}\sin x\,\mathbf{k}$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(e^{-x}\sin y) + \frac{\partial}{\partial y}(e^{-y}\sin z) + \frac{\partial}{\partial z}(e^{-z}\sin x) = -e^{-x}\sin y - e^{-y}\sin z - e^{-z}\sin x$$

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{-x}\sin y & e^{-y}\sin z & e^{-z}\sin x \end{vmatrix}$$

$$= \langle 0 - e^{-y}\cos z, -(e^{-z}\cos x - 0), 0 - e^{-x}\cos y \rangle$$

$$= \langle -e^{-y}\cos z, -e^{-z}\cos x, -e^{-x}\cos y \rangle$$

2. Evaluate the line integral $\int_C x^3 z\,ds$ where C is the curve

$\vec{r}(t) = (\overset{x}{2\sin t})\mathbf{i} + \overset{y}{t}\mathbf{j} + (\overset{z}{2\cos t})\mathbf{k}$ for $0 \le t \le \pi/2$.

$$ds = \|r'(t)\| \qquad r'(t) = \langle 2\cos t, 1, -2\sin t \rangle$$

$$ds = \sqrt{4\cos^2 t + 1 + 4\sin^2 t}$$

$$= \sqrt{4+1} = \sqrt{5}$$

$$f = x^3 z = 8\sin^3 t \cdot 2\cos t$$

$$= 16\sin^3 t\cos t$$

$$\int_C x^3 z\,ds = \int_0^{\pi/2} 16\sin^3 t\cos t\sqrt{5}\,dt = \frac{16}{4}\sin^4 t\sqrt{5}\Big|_0^{\frac{\pi}{2}} = \underline{\underline{4\sqrt{5}}}$$

3. Evaluate $\int_S (x^2 z + y^2 z)\,dS$ where S is part of the plane $z = 4 + x + y$ that lies inside the rectangle $0 \le x \le 1,\ 0 \le y \le 1$.

$$\bar{r}(x,y) = \langle x, y, 4+x+y \rangle$$

$$\bar{r}_x = \langle 1, 0, 1 \rangle$$

$$\bar{r}_y = \langle 0, 1, 1 \rangle$$

$$\bar{r}_x \times \bar{r}_y = \langle -1, -1, 1 \rangle$$

$$dS = \|\bar{r}_x \times \bar{r}_y\| = \sqrt{3}\,dA$$

$$= \int_0^1\int_0^1 \left[x^2(4+x+y) + y^2(4+x+y)\right]\sqrt{3}\cdot dy\,dx$$

4. Use Green's Theorem to evaluate $\int x^2y\,dx - xy^2\,dy$ where C is the circle $x^2+y^2=4$ with counterclockwise orientation.

$$\int F_1\,dx + F_2\,dy = \iint\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(-y^2 - x^2\right)dy\,dx$$

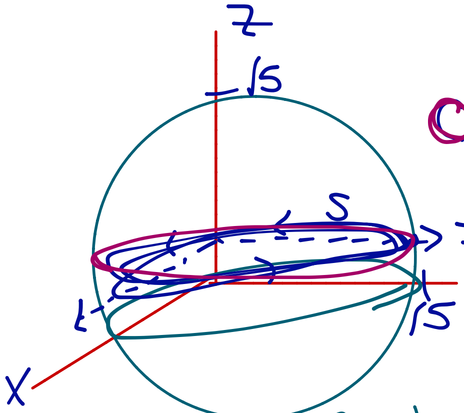
$F_1 = x^2y$

$F_2 = -xy^2$

$$= \int_0^{2\pi}\int_0^{2} -r^2\, r\,dr\,d\theta$$

5. Use Stokes' Theorem to evaluate $\iint_S (curl\,\vec{F})\cdot\vec{n}\,dS$ where $\vec{F}(x,y,z)=\langle x^2yz, yz^2, z^3e^{xy}\rangle$, S is part of the sphere $x^2+y^2+z^2=5$ that lies above the plane $z=1$ and S is oriented upward.

$r(\theta) = \langle 2\cos\theta, 2\sin\theta, 1\rangle$



C is a circle $x^2+y^2=2^2$

a circle of radius 2 on the plane $z=1$

$$\int F\cdot dr = \int \langle 2\cos^2\theta\cdot 2\sin\theta,\ 2\sin\theta,\ e^{2\sin\theta\cdot 2\cos\theta}\rangle\cdot\langle -2\sin\theta, 2\cos\theta, 0\rangle$$

$$= \int_0^{2\pi}\left(-16\cos^2\theta\sin^2\theta + 4\sin\theta\cos\theta + 0\right)d\theta$$

Use power-reducing and half-angle id.

CCW $\langle\cos\theta, \sin\theta\rangle$

CW $\langle\sin\theta, \cos\theta\rangle \to \langle\cos\theta, \sin(-\theta)\rangle$

$= \langle\cos\theta, -\sin\theta\rangle$

6. Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}=(4x^3y^2-2xy^3)\mathbf{i}+(2x^4y-3x^2y^2+4y^3)\mathbf{j}$ and C: $\vec{r}(t)=(t+\sin\pi t)\mathbf{i}+(2t+\cos\pi t)\mathbf{j}$, $0\le t\le 1$.

where f is the potential of F

if $\bar{F}$ is cons. $(\text{curl}\,\bar{F} = \bar{0})$

then $f(B) - f(A) = \int_A^B \nabla f\cdot dr = \int \bar{F}\cdot dr$

$$\text{curl}\,\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4x^3y^2-2xy^3 & 2x^4y-3x^2y^2+4y^3 & 0 \end{vmatrix} = \langle 0, 0, (8x^3y-6xy^2)-(8x^3y-6y^2)\rangle$$

$= \bar{0}$ ✓

where f is the potential of F

If $\bar{F}$ is cons. $(\text{curl}\,\bar{F} = \bar{0})$

then $f(B) - f(A) = \int_A^B \nabla f \cdot dr = \int \bar{F} \cdot dr$

$$\text{curl}\,\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4x^3y^2 - 2xy^3 & 2x^4y - 3x^2y^2 + 4y^3 & 0 \end{vmatrix} = \langle 0, 0, (8x^3y - 6xy^2) - (8x^3y - 6y^2) \rangle$$

$= \bar{0}$ ✓

$$\int (4x^3y^2 - 2xy^3)\,dx = x^4y^2 - x^2y^3 + c$$

$$\int (2x^4y - 3x^2y^2 + 4y^3)\,dy = x^4y^2 - x^2y^3 + y^4 + c$$

$$f(x,y) = x^4y^2 - x^2y^3 + y^4 + c$$

$f(B) - f(A) =$

$r(0) = \langle 0, 1 \rangle$

$r(1) = (1, 1)$

$$f(1,1) - f(0,1) = (1^4 1^2 - 1^2 1^3 + 1^4) - (0 - 0 + 1)$$

$$1 - 1 = 0$$

7. Evaluate the line integral $\int_C (xy + \ln x)\,dy$ where C is the arc of the parabola $y = x^2$ from (1, 1) to (3, 9).

$\vec{F} = \langle xy, \ln x \rangle$

$d\vec{r} = \langle dx, dy \rangle$

$\int_C \vec{F} \cdot d\vec{r} = \int_C xy\,dx + \ln dy$

$\vec{r}(x) = \langle x, x^2 \rangle \quad 1 \le x \le 3$

$d\vec{r} = \langle 1, 2x \rangle dx$

$2x\,dx = dy$

$\vec{F} = \langle x \cdot x^2, \ln x \rangle$

$= \int_1^3 (x^3 + 2x \ln x)\,dx$

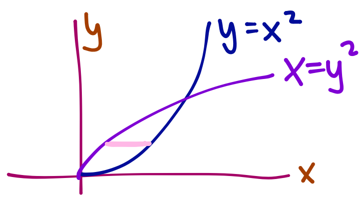
8. Use Green's Theorem to evaluate the line integral $\int_C \left(y + e^{\sqrt{x}}\right)dx + \left(2x + \cos y^2\right)dy$ along the positively oriented curve C where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

$F_2 = 2x + \cos y^2$

$\frac{\partial F_2}{\partial x} = 2$

$\frac{\partial F_1}{\partial y} = 1$

$\iint_C \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = \iint_C (2-1)\,dy\,dx = \int_0^1 \int_{x^2}^{\sqrt{x}} 1 \cdot dy\,dx$



$= \int_0^1 \int_{y^2}^{\sqrt{y}} 1 \cdot dx\,dy$

9. Show that the vector field $\vec{F}(x,y,z) = \left(2xz + y^2\right)\mathbf{i} + 2xy\,\mathbf{j} + \left(x^2 + 3z^2\right)\mathbf{k}$ is conservative and evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C: $x = t^2$, $y = t + 1$, $z = 2t - 1$ for $0 \le t \le 1$.

curves - 1 var.
surfaces - 2 var.

$\int_A^B \nabla f\,d\vec{r} = f(B) - f(A)$

$\text{curl}\,\vec{F} = \vec{0}$

$r(t) = \langle \underset{x}{t^2}, \underset{y}{t+1}, \underset{z}{2t-1} \rangle$

$\vec{F} = \nabla f$ ↖ solve for.

curl $\vec{F} = \vec{0}$ if conservative

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xz+y^2 & 2xy & x^2+3z^2 \end{vmatrix} = \langle 0, -(2x-2x), 2y-2y \rangle = \vec{0} \checkmark$$

$\int (2xz + y^2)\,dx \;\Big|\; \int 2xy\,dy \;\Big|\; \int (x^2 + 3z^2)\,dz$

$= x^2 z + xy^2 + C \;\Big|\; = xy^2 + C \;\Big|_3 = x^2 z + z^3 + C$

$\therefore f = x^2 z + xy^2 + z^3 + C$

$\int \vec{F} \cdot d\vec{r} = \int_{r(0) = A = (0,1,-1)}^{\vec{r}(1) = B = \langle 1,2,1 \rangle} \nabla f\,d\vec{r}$

$= f\langle 1,2,1 \rangle - f\langle 0,1,-1 \rangle = (1^2 \cdot 1 + 1 \cdot 4 + 1^3) - (-1 \cdot 0^2 \cdots) = 7$

10. Evaluate the surface integral $\int_S (x^2+y^2)\,dS$ where S is the surface $z = xy$ inside $x^2+y^2=4$ for $x \ge 0$, $y \ge 0$.

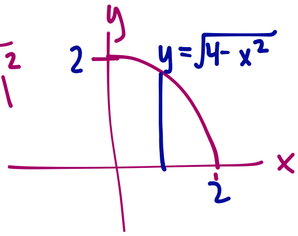
$\vec{r}(x,y) = \langle x, y, xy \rangle$

$$\vec{r}_x = \langle 1, 0, y \rangle \quad (i \; j \; k)$$

$$\vec{r}_y = \langle 0, 1, x \rangle$$

$$\vec{r}_x \times \vec{r}_y = \langle -y, -x, 1 \rangle$$

$$dS = \|\vec{r}_x \times \vec{r}_y\| = \sqrt{(-y)^2+(-x)^2+1^2} = \sqrt{y^2+x^2+1}$$



$$\int_S (x^2+y^2)\,dS = \int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)\sqrt{x^2+y^2+1}\;dy\,dx$$

$$= \int_0^{\pi/2}\int_0^2 r^2\sqrt{r^2+1}\; r\,dr\,d\theta$$

$u = r^2+1$

$du = 2r\,dr$

$$= \int_0^{\pi/2} \frac{1}{2}\int_1^5 (u^2-1)\sqrt{u}\;du$$

11. Evaluate $\iint_S curl\,\vec{F}\cdot\vec{n}\,dS$ where $\vec{F}(x,y,z) = x\,\mathbf{i} + y^2\,\mathbf{j} + xyz\,\mathbf{k}$ and S is part of the paraboloid $z = 4 - x^2 - y^2$ with $z \ge 0$. Use the upward unit normal vector.

Using Stokes' thm $\rightarrow$ the boundary curve is $z = 0 = 4 - x^2 - y^2$

$\Rightarrow 4 = x^2 + y^2 \Rightarrow \vec{r}(\theta) = \langle 2\cos\theta, 2\sin\theta, 0 \rangle \Leftarrow$ CCW

$\vec{F} = \langle 2\cos\theta, 4\sin^2\theta, 0 \rangle \quad d\vec{r} = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$

$\vec{F}\cdot d\vec{r} = -4\sin\theta\cos\theta + 8\sin^2\theta\cos\theta$ So $\int F\cdot dr = \int_0^{2\pi} (-4\sin\theta\cos\theta + 8\sin^2\theta\cos\theta)\,d\theta$

$$= \frac{4}{2}\cos^2\theta\Big|_0^{2\pi} + \frac{8}{3}\sin^3\theta\Big|_0^{2\pi} = 0$$

12. Find the surface area of the cap cut from the paraboloid $y^2+z^2=3x$ by the plane $x=1$.

Surface area $= \int 1 \cdot dS$

$\vec{r}(y,z) = \langle \frac{1}{3}(y^2+z^2), y, z \rangle$

$\frac{1}{3}y^2 + \frac{1}{3}z^2 = x$

$1 = x$

$\vec{r}_y = \langle \frac{2}{3}y, 1, 0 \rangle$

So $1 = \frac{1}{3}y^2 + \frac{1}{3}z^2$

$\vec{r}_z = \langle \frac{2}{3}z, 0, 1 \rangle$

$3 = y^2 + z^2$

$\vec{r}_y \times \vec{r}_z = \langle 1, -\frac{2}{3}y, -\frac{2}{3}z \rangle$

$$dS = \|\vec{r}_y \times \vec{r}_z\| = \sqrt{1 + \frac{4}{9}y^2 + \frac{4}{9}z^2} \quad = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} \sqrt{1+\frac{4}{9}y^2+\frac{4}{9}z^2}\;dz\,dy = \int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{1+\frac{4}{9}r^2}\;r\,dr\,d\theta$$

$ds = \|\bar{r}'(t)\|\, dt$

$\int 1 \cdot ds \Rightarrow$ arc length

$\int 1 \cdot dS \Rightarrow$ surface area

$\bar{F} \cdot \bar{T} d\bar{s} = \bar{F} \cdot d\bar{r}$ $\quad \bar{T} d\bar{s} = \dfrac{\bar{r}'(t)}{\|\bar{r}'(t)\|} \cdot \|\bar{r}'\|\, dt = d\bar{r}$

$= r'(t) dt$

$\int_C f\, ds$ vs $\int_C \bar{F} \cdot d\bar{r} = \int_C \bar{F} \cdot \bar{T} ds$

$dS = \|\bar{r}_u \times \bar{r}_v\|\, dA$

$\bar{n} dS = \dfrac{\bar{r}_u \times \bar{r}_v}{\|\bar{r}_u \times \bar{r}_v\|} \cdot \|\bar{r}_u \times \bar{r}_v\| dA = \bar{r}_u \times \bar{r}_v\, dA$

$\int_S \bar{F} \cdot \bar{n} dS$

$\int_S \bar{F} \cdot dS$ flux through surface

$\int_C \bar{F}\, d\bar{r} = \int_C \bar{F} \cdot \bar{T} ds$

$\int_S \bar{F} \cdot \bar{n} dS$ flux in direction of normal