1. Given the position function of an object $\vec{r}(t)=t^3\mathbf{i}+6t^2\mathbf{j}+\sin t\mathbf{k}$ with $0\le t\le\pi$ find the velocity and acceleration vectors.

$\vec{r} = \langle t^3, t^2, \sin t \rangle$

$\vec{v}(t) = \langle 3t^2, 2t, \cos t \rangle$

$\vec{a}(t) = \langle 6t, 2, -\sin t \rangle$

2. Let $\vec{u}=\langle 3,-2,-1\rangle$ and $\vec{v}=\langle 5,-3,-5\rangle$

a. Find a unit vector in the direction of $\vec{u}$

$\|\vec{u}\| = \sqrt{9+4+1} = \sqrt{14}$

$\boxed{\frac{1}{\sqrt{14}}\langle 3,-2,-1\rangle = \left\langle \frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}} \right\rangle}$

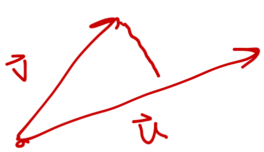
b. Find $\vec{u}\cdot\vec{v}$.

$\vec{u}\cdot\vec{v} = 3(5)+(-2)(-3)+(-1)(-5) = 15+6+5 = \boxed{26}$

c. Find $\vec{u}\times\vec{v}$.

$$\vec{u}\times\vec{v} = \begin{vmatrix} \overset{+}{i} & \overset{-}{j} & \overset{+}{k} \\ 3 & -2 & -1 \\ 5 & -3 & -5 \end{vmatrix} = i(10-3) - j(-15+5) + k(-9+10) = \boxed{7i + 10j + k}$$

d. Find the projection of $\vec{v}$ onto $\vec{u}$.



$\text{proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|}\right)\frac{\vec{u}}{\|\vec{u}\|} = \frac{26}{14}\langle 3,-2,-1\rangle$

$= \boxed{\frac{13}{7}\langle 3,-2,-1\rangle}$

e. Find the angle between $\vec{u}$ and $\vec{v}$.

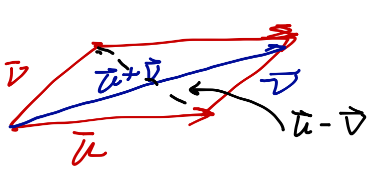
$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{26}{\sqrt{14}\sqrt{59}} \Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{26}{\sqrt{14}\sqrt{59}}\right)}$

f. Find a vector parallel to $\vec{v}$ with length 5 units.

$\frac{5}{\sqrt{59}}\langle 5,-3,-5\rangle = \boxed{\left\langle \frac{25}{\sqrt{59}}, \frac{-15}{\sqrt{59}}, \frac{-25}{\sqrt{59}} \right\rangle}$

g. Compute the lengths of the diagonals of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

$\vec{u}+\vec{v} = \langle 3,-2,-1\rangle + \langle 5,-3,-5\rangle = \boxed{\langle 8,-5,-6\rangle}$



$\vec{u}-\vec{v} = \langle 3,-2,-1\rangle + \langle -5,3,5\rangle = \boxed{\langle -2,1,4\rangle}$

h. Find a vector orthogonal to both $\vec{u}$ and $\vec{v}$.

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & -1 \\ 5 & -3 & -5 \end{vmatrix} = i(10-3) - j(-15+5) + k(-9+10) = \boxed{7\vec{i} + 10\vec{j} + \vec{k}}$$

i. Compute the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

$$\|\vec{u}\times\vec{v}\| = \sqrt{49+100+1} = \boxed{\sqrt{150}}$$

2. Find a vector equation of the line that passes through the points $(2,6,-1)$ and $(-6,4,0)$.

$\vec{v} = \langle 8, 2, -1\rangle$ $\boxed{\vec{r}(t) = \langle 2,6,-1\rangle + t\langle 8, 2, -1\rangle \quad t\in\mathbb{R}}$

3. Find the parametric equations of the line that passes through the point $(0,1,1)$ and parallel to the vector $\langle 2,-1,3\rangle$.

$\boxed{x = 2t, \quad y = 1-t \quad z = 1+3t \quad t\in\mathbb{R}}$

4. Find a vector equation of the line that passes through $(2,1,0)$ and perpendicular to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{j}+\mathbf{k}$.

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \vec{i}(1-0) - j(1-0) + \vec{k}(1-0) = \langle 1,-1,1\rangle$$

$\boxed{\vec{r}(t) = \langle 2,1,0\rangle + t\langle 1,-1,0\rangle \quad t\in\mathbb{R}}$

5. Find the equation of the plane through $A(-1,6,-5)$ and parallel to the plane $x+y+z+2=0$.

$\vec{N} = \langle 1,1,1\rangle$ $\langle 1,1,1\rangle\cdot\langle x+1, y-6, z+5\rangle = 0$

$\Rightarrow x+1+y-6+z+5=0$

$\Rightarrow \boxed{x+y+z=0}$

6. Find an equation of the plane through $P(-2,3,1)$, $Q(1,1,0)$, and $R(-1,0,1)$.

$\vec{PQ} = \langle 3, -2, -1 \rangle$

$\vec{PR} = \langle 1, -3, 0 \rangle$

$\vec{N} = \vec{PQ} \times \vec{PR} = \langle -3, -1, -7 \rangle$

Plane: $-1\langle 3, 1, 7\rangle \cdot \langle x+2, y-3, z-1\rangle = 0$

$\Rightarrow 3(x+2) + (y-3) + 7(z-1) = 0$

$\Rightarrow \boxed{3x + y + 7z = 4}$

7. Find the equation of the sphere with center $(2,-3,4)$ and tangent to the *xz*-plane.

xz-plane is $y=0$

radius is how far from pt $(2,-3,4)$ to plane $y=0$ which is 3 units.

Sphere: $\boxed{(x-2)^2 + (y+3)^2 + (z-4)^2 = 9}$

8. Find the equation of the sphere with the line segment $(1,5,-2)$ and $(3,9,-1)$ is the diameter.

Center is midpoint of the diameter

$x = \frac{1+3}{2} = 2$, $y = \frac{5+9}{2} = 7$ $z = \frac{-2-1}{2} = \frac{-3}{2}$

$(2, 7, -3/2)$

radius is $\frac{1}{2}$ distance of diameter.

$\frac{1}{2}\sqrt{(3-1)^2 + (9-5)^2 + (-1+2)^2} = \frac{1}{2}\sqrt{4+16+1} = \frac{\sqrt{21}}{2}$

$\boxed{\text{Sphere: } (x-2)^2 + (y-7)^2 + (z+3/2)^2 = \frac{21}{4}}$

9. Find the distance between the points $P(1,2,-3)$ and $Q(4,7,-3)$.

$\vec{PQ} = \langle 3, 5, 0 \rangle$ $\|\vec{PQ}\| = \sqrt{9+25} = \boxed{\sqrt{34}}$

10. Find an equation of the line of intersection of the two plane $y - 5z = 3$ and $6x - 7y = 5$.

$\vec{N}_1 = \langle 0, 1, -5 \rangle$ $\vec{N}_2 = \langle 6, -7, 0 \rangle$

$$\vec{V} = \vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} i & j & k \\ 0 & 1 & -5 \\ 6 & -7 & 0 \end{vmatrix} = \langle -35, -30, -6 \rangle$$

$x = \frac{13}{3}$, $y = 3$, $z = 0$ ← pt of intersection (see next page)

Line

$\boxed{r(t) = \langle \tfrac{13}{3}, 3, 0 \rangle + t\langle -35, -30, -6 \rangle \quad t \in \mathbb{R}}$

11. Find an equation of the line tangent to the curve $\vec{r}(t) = \langle \sin 2t, \cos 2t, t \rangle$ at $t = \dfrac{\pi}{4}$

$\vec{r}'(t) = \langle 2\cos 2t, -2\sin 2t, 1 \rangle$

$\vec{r}(\frac{\pi}{4}) = \langle 1, 0, \pi/4 \rangle$

$\vec{r}'(\pi/4) = \langle 2\cos \pi/2, -2\sin \pi/2, 1 \rangle$

$= \langle 0, -2, 1 \rangle$

Line:

$\boxed{\vec{r}(t) = \langle 1, 0, \pi/4 \rangle + t\langle 0, -2, 1 \rangle}$

planes $6x - 7y = 5$ & $y - 5z = 3$

Find pt of intersection:

$$\begin{array}{r} 6x - 7y + 0z = 5 \\ +\ \ 0x + 7y - 35z = 21 \\ \hline 6x - 35z = 26 \end{array}$$ $\leftarrow$ let $z=0$

$6x = 26 \Rightarrow x = \frac{13}{3}$

& if $z=0$ $y - 5z = 3 \Rightarrow y = 3$

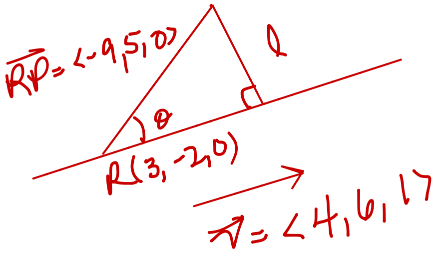
Check $6\left(\frac{13}{3}\right) - 7(3) = 5$ ✓

pt of intersection $(13/3, 3, 0)$

$\vec{RP} = \langle -9, 5, 0\rangle$ $\quad \vec{RP}\times\vec{v} = \langle 5, +9, -74\rangle$

$\vec{v} = \langle 4, 6, 1\rangle$

12. Determine the distance from the point $P(-6,3,0)$ to the line given by $\vec{r}(t) = \langle 3+4t, -2+6t, t\rangle$.



$\frac{l}{\|\vec{RP}\|} = \sin\theta \Rightarrow l = \|\vec{RP}\|\sin\theta$ now $\|\vec{RP}\times\vec{v}\| = \|\vec{RP}\|\|\vec{v}\|\sin\theta$

so $l = \frac{\|\vec{RP}\times\vec{v}\|}{\|\vec{v}\|} = \boxed{\frac{\sqrt{25+81+(74)^2}}{\sqrt{16+36+1}}}$

13. Show that the lines with equation

$$\frac{x+1}{2} = \frac{y-3}{-4} = \frac{z-2}{5} \text{ and } \frac{x-4}{3} = \frac{y+1}{2} = \frac{z}{3}$$

are skew, fine the equations of the parallel planes containing the lines, and find the distance between the lines.

$\vec{v}_1 = \langle 2, -4, 5\rangle$ Since $\vec{v}_1 \neq k\vec{v}_2$ the lines are not parallel.

$\vec{v}_2 = \langle 3, 2, 3\rangle$

x: $2t-1 = 3s+4$
y: $-4t+3 = 2s-1$
z: $5t+2 = 3s$

$\left.\right\}$ solve for t & s

$2t = 3s+5$

$t = \frac{3}{2}s + \frac{5}{2}$

$-4\left(\frac{3}{2}s+\frac{5}{2}\right)+3 = 2s-1$

$-6s-10+3 = 2s-1$

$-8s = -6$

$s = 6/8 = 3/4$

$t = \frac{3}{2}\left(\frac{3}{4}\right)+\frac{5}{2}$

$= \frac{9}{8}+\frac{20}{8} = \frac{29}{8}$

Check $5\left(\frac{29}{8}\right)+2 = 3\left(\frac{3}{4}\right)$

$\frac{145}{8}+2 \neq \frac{9}{4}$ $\therefore$ The 2 lines do not intersect & are skew

see next page for the rest of the solution

14. Let $\mathbf{r}(t) = \left\langle \sqrt{2-t}, (e^t-1)/t, \ln(t+1)\right\rangle$.

a. Find the domain of **r**.
b. Find $\lim_{t\to 0}\vec{r}(t)$

a) $\sqrt{2-t} \leftarrow t \le 2$

$\frac{e^t-1}{t}$ any real except $t=0$

$\ln(t+1)$ $t > -1$

$\boxed{\text{Domain of } \vec{r}(t) \quad (-1,0)\cup(0,2]}$

b) $\lim_{t\to 0}\vec{r}(t) = \left\langle \lim_{t\to 0}\sqrt{2-t}, \lim_{t\to 0}\frac{e^t-1}{t}, \lim_{t\to 0}\ln(t+1)\right\rangle = \boxed{\langle\sqrt{2}, 1, 0\rangle}$

$\vec{v}_1 = \langle 2, -4, 5 \rangle$ $\quad \vec{N} = \vec{v}_1 \times \vec{v}_2 = \langle -22, 9, 16 \rangle$

$\vec{v}_2 = \langle 3, 2, 3 \rangle$

$x: \ 2t - 1 = 3s + 4$ $\quad P_1(-1, 3, 2) \quad P_2(4, -1, 0)$

$y: \ -4t + 3 = 2s - 1$

$z: \ 5t + 2 = 3s$

$$\boxed{\begin{array}{l}\text{Planes:}\\ -22(x+1) + 9(y-3) + 16(z-2) = 0 \\ -22(x-4) + 9(y+1) + 16z = 0\end{array}}$$

Distance Between 2 planes

$\overrightarrow{P_1P_2} = \langle 5, -4, -2 \rangle$

$$l = \frac{|\vec{N} \cdot \overrightarrow{P_1P_2}|}{\|\vec{N}\|} = \frac{-110 + (-36) + -32}{\sqrt{484 + 81 + 256}}$$

$$\boxed{= \frac{178}{\sqrt{821}}}$$

15. Given the points A(1, 0 ,1), B(2, 3, 0) and C(-1, 1, 4), find the area of the triangle with vertices A, B, and C.

$\vec{AB} = \langle 1, 3, -1 \rangle$

$\vec{AC} = \langle -2, 1, 3 \rangle$

$\vec{AB} \times \vec{AC} = \langle 10, -1, 7 \rangle$

Area of triangle

$\frac{1}{2} \| \vec{AB} \times \vec{AC} \| = \boxed{\frac{1}{2}\sqrt{150}}$

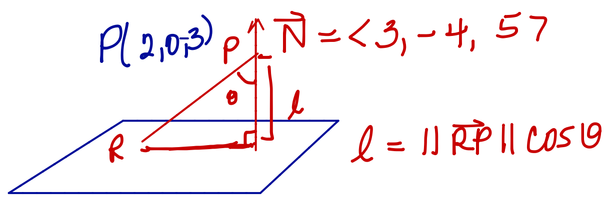
16. Find the length of the curve $\vec{r}(t) = \langle 2t^{3/2}, \cos 2t, \sin 2t \rangle$ for $0 \le t \le 1$.

$\vec{r}\,'(t) = \langle 3t^{1/2}, -2\sin 2t, 2\cos 2t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{9t + 4\sin^2 2t + 4\cos^2 2t} = \sqrt{9t+4}$

$L = \int_0^1 \sqrt{9t+4}\, dt = \frac{1}{9} \cdot \frac{2}{3} (9t+4)^{3/2} \Big|_0^1 = \boxed{\frac{2}{27}\left[13^{3/2} - 8\right]}$

17. Find the distance from the point $(2,0,-3)$ to the plane $3x - 4y + 5z = 1$.



R(1/3, 0, 0)

$\vec{RP} = \langle 5/3, 0, -3 \rangle$

Def: $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

So $l = \dfrac{|N \cdot RP|}{\|N\|} = \dfrac{|5 + 0 + (-15)|}{\sqrt{9+16+25}}$

$= \dfrac{10}{\sqrt{50}} = \dfrac{10}{5\sqrt{2}} = \dfrac{2}{\sqrt{2}} = \boxed{\sqrt{2}}$

18. A particle moves in space with parametric equations $x = t, y = t^2, z = \frac{4}{3}t^{3/2}$. Find the curvature of its trajectory, the unit tangent vector, and the unit normal vector when $t = 1$.

$\vec{r}\,' = \langle 1, 2t, 2t^{1/2} \rangle$

$\|\vec{r}\,'\| = \sqrt{1 + 4t^2 + 4t}$

$= 1 + 2t$

$\vec{T}(t) = \frac{1}{1+2t} \langle 1, 2t, 2t^{1/2} \rangle$

$\boxed{\vec{T}(1) = \frac{1}{3}\langle 1, 2, 2 \rangle = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle}$

$\frac{dT}{dt} = -1(1+2t)^{-2}(2)\langle 1, 2t, 2t^{1/2} \rangle +$

$\frac{1}{1+2t} \langle 0, 2, t^{-1/2} \rangle$

$\frac{dT}{dt}\Big|_{t=1} = \frac{-2}{9}\langle 1, 2, 2 \rangle + \frac{1}{3}\langle 0, 2, 1 \rangle$

$= \langle -\frac{2}{9}, -\frac{4}{9}, -\frac{4}{9} \rangle + \langle 0, \frac{2}{3}, \frac{1}{3} \rangle = \langle -\frac{2}{9}, \frac{2}{9}, -\frac{1}{9} \rangle$

$\vec{N}(1) = \dfrac{\langle -\frac{2}{9}, \frac{2}{9}, -\frac{1}{9} \rangle}{\sqrt{4/81 + 4/81 + 1/81}}$

$\boxed{= \langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \rangle}$

$\boxed{K(1) = \dfrac{\|T'(1)\|}{\|r'(1)\|} = \dfrac{1/3}{3} = \dfrac{1}{9}}$

$P(1, 1, 4/3)$

19. Find the binormal vector and osculating plane for $x=t, y=t^2, z=\frac{4}{3}t^{3/2}$ at $t=1$.

$\vec{B}=\vec{T}\times\vec{N}$

$\boxed{\vec{B} = \langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \rangle}$

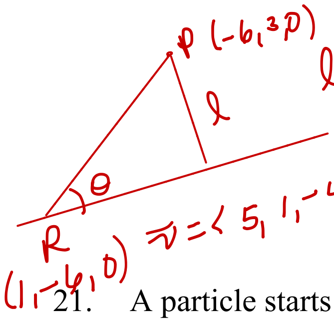
$\vec{T} = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$

$\vec{N} = \langle -\frac{2}{3}, \frac{2}{3}, \frac{-1}{3} \rangle$

Osc. Plane

$\boxed{-\frac{2}{3}(x-1) - \frac{1}{3}(y-1) + \frac{2}{3}(z-\frac{4}{3}) = 0}$

20. Find the distance from the point $(-6,3,0)$ to the line given by $\vec{r}(t)=\langle 1+5t,-6+t,-4t\rangle$



$\ell = \|RP\| \sin\theta$

$\vec{v} = \langle 5, 1, -4 \rangle$

$\vec{RP} = \langle -7, 9, 0 \rangle$

$\frac{\|\vec{RP}\times\vec{v}\|}{\|\vec{v}\|} = \ell$

$\boxed{\ell = \frac{4\sqrt{81+49+169}}{\sqrt{25+1+16}}}$

$\begin{vmatrix} i & j & k \\ -7 & 9 & 0 \\ 5 & 1 & -4 \end{vmatrix}$

$= \langle -36, -28, -52 \rangle$

$= -4\langle 9, 7, 13 \rangle$

21. A particle starts at the origin with initial velocity $\mathbf{i}-\mathbf{j}+3\mathbf{k}$. Its acceleration is $\mathbf{a}(t)=6t\,\mathbf{i}+12t^2\mathbf{j}-6t\,\mathbf{k}$. Find its position function.

$\vec{v}(t) = \int a(t)\,dt$

$= \langle 3t^2+C_1, 4t^3+C_2, -3t^2+C_3 \rangle$

$v(0) = \langle C_1, C_2, C_3 \rangle = \langle 1, -1, 3 \rangle$

$v(t) = \langle 3t^2+1, 4t^3-1, -3t^2+3 \rangle$

$\vec{r}(t) = \int \vec{v}(t)\,dt$

$= \langle t^3+t+\cancel{D_1}^{0}, t^4-t+\cancel{D_2}^{0}, -t^3+3t+\cancel{D_3}^{0} \rangle$

$\vec{r}(0)=\vec{0}$

$\boxed{\vec{r}(t) = \langle t^3+t, t^4-t, -t^3+3t \rangle}$

22. Find the tangential and normal components of acceleration for a particle moving along a conical helix defined by $\vec{r}(t)=\langle t\cos t, t\sin t, t\rangle$.

23. The helix $\mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ intersects the curve $\mathbf{r}_2(t) = (1+t)\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ at point (1, 0, 0). Find the angle of intersection of these curves.

$\vec{r}_1{}' = \langle -\sin t, \cos t, 1 \rangle$ $\quad \vec{r}_1{}'(0) = \langle 0, 1, 1 \rangle$ $\quad \cos\theta = \dfrac{\vec{r}_1{}' \cdot \vec{r}_2{}'}{\|\vec{r}_1{}'\| \, \|\vec{r}_2{}'\|}$

$\vec{r}_2{}' = \langle 1, 2t, 3t^2 \rangle$ $\quad \vec{r}_2{}'(0) = \langle 1, 0, 0 \rangle$

$\vec{r}_1{}' \cdot \vec{r}_2{}' = 0$ Thus the angle of intersection is $\frac{\pi}{2}$.

For additional problems, check out the review problems for Chapters 10 and 11. Note the questions above are simply a sample of questions possible for the exam; it is possible that other types of questions may appear on your exam.

$\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}$