Exam 4 **Name**______________________________

Math 231-01 **Calculus III** **August 2, 2018**

Guidelines

- **Calculators are not allowed.**
- Read the questions carefully. You have 65 minutes; use your time wisely.
- You may leave your answers in symbolic form, like $\sqrt{3}$ or $\ln(2)$, unless they simplify further like $\sqrt{9} = 3$ or $\cos(3\pi/4) = -\sqrt{2}/2$.
- Put a box around your final answers when relevant.
- Show all steps in your solutions and make your reasoning clear. Answers with no explanation will not receive full credit, even when correct.
- Use the space provided. If necessary, write "see other side" and continue working on the back of the same page.
- $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos$ and $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin$
- $x = \rho \sin . \cos$, $y = \rho \sin \sin$, $z = \rho \cos$, and $dV = \rho^2 \sin d\rho\, d\varphi\, d\theta$
- **Green's Theorem** Let C be a closed bounded curve that bounds a region R in the plane and let $\mathbf{F}(x, y)) = f\,\mathbf{i} + g\,\mathbf{j}$ be a vector field then
  1. Circulation: $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.$
  2. Outward Flux Integral: $\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = \oint_C f\,dy - g\,dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA$
- **Stokes' Theorem** Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary C whose orientation is consistent with that of S. Assume that $\mathrm{F} = \langle f, g, h\rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

  $$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

  when $\mathfrak{n}$ is the unit vector normal to S determined by the orientation of S.

Question	Points	Score
1	8	
2	10	
3	10	
4	10	
5	10	
6	10	
7	12	
8	10	
9	10	
10	10	
Total:	100	

1. (8 points) Complete test corrections.

2. (10 points) Evaluate the line integral $\int_C (y-z)\,ds$ where C is the helix $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, t\rangle$ for $0 \le t \le 2\pi$.

Solution:

Given $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, t\rangle$ then
$ds = |\mathbf{r}'|dt = |\langle -3\sin t, 3\cos t, 1\rangle|dt = \sqrt{9\sin^2 t + 9\cos^2 t + 1}\,dt = \sqrt{10}\,dt$

$$\begin{aligned}\int_C (y-z)\,ds &= \int_0^{2\pi} (3\sin t - t)\sqrt{10}\,dt \\ &= \sqrt{10}\left(-3\cos t\Big|_0^{2\pi} - \frac{1}{2}t^2\Big|_0^{2\pi}\right) = \sqrt{10}\left(-3(1-1) - \frac{1}{2}4\pi^2\right) \\ &= -2\sqrt{10}\,\pi\end{aligned}$$

3. (10 points) Compute the divergence and curl of the vector field $\mathbf{F} = \langle 2xy + z^4, x^2, 4xz^3\rangle$. State whether the field is source free or irrotational.

Solution:

The divergence is $\nabla \cdot \mathbf{F} = \dfrac{\partial(2xy+z^4)}{\partial x} + \dfrac{\partial(x^2)}{\partial y} + \dfrac{\partial(4xz^3)}{\partial z} = 2y + 12xz^2$.

The curl is $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy+z^4 & x^2 & 4xz^3 \end{vmatrix} = \langle 0, 0, 0\rangle$

The field is not source free $\nabla \cdot \mathbf{F} \neq 0$; however, it is irrotational becauase $\nabla \times \mathbf{F} = \mathbf{0}$.

4. (10 points) Use a surface integral to find the area of the surface of a frustum of the cone $z^2 = x^2 + y^2$ for $2 \le z \le 4$ (excluding the bases).

Solution:

So the surface is part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 4$

The parameterization of the surface is $\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r\rangle$ where $2 \le r \le 2\pi$, $0 \le \theta \le 2\pi$.

First $dS = |\mathbf{r}_r \times \mathbf{r}_\theta| dA$

So $\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r\rangle$

Thus $dS = |\mathbf{r}_r \times \mathbf{r}_\theta| dA = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2}\, dA = r\sqrt{2}\, dA$

$$\begin{aligned}\iint_S dS &= \int_0^{2\pi} \int_2^4 r\sqrt{2}\, dr\, d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{2} r^2 \Big|_2^4 d\theta \\ &= \int_0^{2\pi} \frac{\sqrt{2}}{2}(16 - 4)\, d\theta = 6\sqrt{2}\theta \Big|_0^{2\pi} \\ &= 12\sqrt{2}\,\pi\end{aligned}$$

5. (10 points) Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ using Stokes' Theorem where $\mathbf{F} = \langle x^2 - y^2, x, 2yz\rangle$; C is the boundary of the plane $z = 6 - x - y$ in the first octant. Assume that C has counterclockwise orientation. Set up but do not evaluate.

Solution:

Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$

First the curl is

$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & x & 2yz \end{vmatrix} = \langle 2z - 0, -(0 - 0), 1 + 2y\rangle = \langle 2z, 0, 1 + 2y\rangle$

Second $\mathbf{n}\, dS = \dfrac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} |\mathbf{r}_x \times \mathbf{r}_y| dA = \mathbf{r}_x \times \mathbf{r}_y\, dA$

$\mathbf{r}(x, y) = \langle x, y, 6 - x - y\rangle$ for $0 \le x \le 6$, $0 \le y \le 6 - x$

So $\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1\rangle$

Thus $(\nabla \times \mathbf{F}) \cdot \mathbf{r}_x \times \mathbf{r}_y \, dA = (2z + 1 + 2y) \, dA$

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \int_0^6 \int_0^{6-x} (2y + 2z + 1) \, dy \, dx$$

6. (10 points) Evaluate $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$, where C is the circle $x^2 + y^2 = 9$.

Solution:

Use Green's Theorem (Circulation)

$\mathbf{F} = \langle f, g\rangle = \left\langle 3y - e^{\sin x}, 7x + \sqrt{y^4+1}\right\rangle$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 7 - 3 = 4$$

$$\begin{aligned}\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy &= \iint \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\,dA\\ &= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4)\ dy\,dx\\ &= \int_0^{2\pi}\int_0^3 4r\,dr\,d\theta = \int_0^{2\pi} 2r^2\Big|_0^3\,d\theta\\ &= 54\theta\Big|_0^{2\pi} = 108\pi\end{aligned}$$

7. Given $\mathbf{F} = \langle y^2z + 2xz^2, 2xyz, xy^2 + 2x^2z\rangle$ and C is $\mathbf{r}(t)\left\langle\sqrt{t}, t+1, t^2\right\rangle$ for $0 \le t \le 1$.

a. (6 points) Find a function φ such that $\mathbf{F} = \nabla\varphi$

Solution:

$\mathbf{F} = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z\rangle$

To find $\varphi(x, y, z)$ start by integrating each partial derivative of $\nabla\varphi$ with respect to the variable list.

$\int \varphi_x\,dx = \int (y^2z + 2xz^2)\,dx = xy^2z + x^2z^2$

$\int \varphi_y\,dy = \int (2xyz)\,dy = xy^2z$

$\int \varphi_z\,dz = \int (xy^2 + 2x^2z)\,dz = xy^2z + x^2z^2$

Combining the three antiderivative we get

$$\varphi(x, y, z) = xy^2z + x^2z^2 + C.$$

(Each element in the three anitderivatives is added only once.)

b. (6 points) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$.

Solution:

Since **F** is conservative, it is path independent

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$$

where B is the point associated with $\mathbf{r}(1) = \left\langle \sqrt{1}, 1+1, 1^2 \right\rangle = \langle 1, 2, 1 \rangle$ so B is the point $(1, 2, 1)$ and A is the point associated with $\mathbf{r}(0) = \langle 0, 1, 0 \rangle$ so A is the point $(0, 1, 0)$

Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(1, 2, 1) - \varphi(0, 1, 0) = 4 + 1 - 0 - 0 = 5$$

8. (10 points) Use Stokes' Theorem to evaluate the surface integral $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$ where $\mathbf{F} = \langle x^2 - y^2, y^2, xz\rangle$; S is the hemisphere $x^2 + y^2 + z^2 = 4$ for $y \geq 0$. The normal vector is outward, points to positive y-axis.

Solution:

Evaluate $\oint \mathbf{F}\cdot d\mathbf{r}$

Note: when $y = 0$ we get $x^2 + z^2 = 4 \implies r = 2$ so C is a circle of radius 2 centered at the origin. So C is $\mathbf{r}(\theta) = \langle 2\cos\theta, 0, 2\sin\theta\rangle$ for $0 \leq \theta \leq 2\pi$. This circle is traversed counterclockwise, assuming the the normal vector for the surface was upward.

Now $d\mathbf{r} = \langle -2\sin\theta, 0, 2\cos\theta\rangle$ and
$\mathbf{F} = \langle x^2 - y^2, y^2, xz\rangle = \langle 4\cos^2\theta, 0, 4\cos\theta\sin\theta\rangle$

$$\oint \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \left(-8\cos^2\theta\sin\theta + 8\cos^2\theta\sin\theta\right)\,d\theta = 0$$

9. (10 points) Evaluate the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ for $\mathbf{F} = \langle x, y, z^2\rangle$; S is the sphere with radius 1 and center at the origin. Set up, but do not evaluate.

Solution:

First $\mathbf{n}\,dS = \dfrac{\mathbf{r}_\varphi\times\mathbf{r}_\theta}{|\mathbf{r}_\varphi\times\mathbf{r}_\theta|}|\mathbf{r}_\varphi\times\mathbf{r}_\theta|dA = \mathbf{r}_\varphi\times\mathbf{r}_\theta\,dA$

$\mathbf{r}(\varphi,\theta) = \langle\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi\rangle$ for $0 \leq \varphi \leq \pi$, $0 \leq \theta \leq 2\pi$

So

$$\begin{aligned}\mathbf{r}_\varphi\times\mathbf{r}_\theta &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi\\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0\end{vmatrix}\\ &= \left\langle\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \cos\varphi\sin\varphi\cos^2\theta + \cos\varphi\sin\varphi\sin^2\theta\right\rangle\\ &= \left\langle\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \cos\varphi\sin\varphi\right\rangle\end{aligned}$$

Second, $\mathbf{F} = \langle x, y, z^2\rangle = \langle\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos^2\varphi\rangle$

Thus
$\mathbf{F}\cdot\mathbf{r}_r\times\mathbf{r}_\theta\,dA = (\sin^3\varphi\cos^2\theta + \sin^3\varphi\sin^2\theta + \cos^3\varphi\sin\varphi)\,dA = (\sin^3\varphi + \cos^3\varphi\sin\varphi)\,dA$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^{2\pi} \int_0^{\pi} (\sin^3 \varphi + \cos^3 \varphi \sin \varphi)\, d\varphi\, d\theta$$

10. (10 points) Evaluate the line integral $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ for $\mathbf{F} = \langle xy^2, -x^2 \rangle$; C is $\mathbf{r}(t) = \langle t^3, t^2 \rangle$ for $0 \leq t \leq 1$.

Solution:

First $\mathbf{F} = \langle xy^2, -x^2 \rangle = \langle t^3(t^2)^2, -(t^3)^2 \rangle = \langle t^7, -t^6 \rangle$

and $d\mathbf{r} = \langle 3t^2, -6t^5 \rangle \, dt$.

$\mathbf{F} \cdot d\mathbf{r} = (3t^9 + 6t^{11}) \, dt$.

Thus $\int \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (3t^9 + 6t^{11} \, dt = \left(\frac{3}{10} t^{10} + \frac{6}{12} t^{12} \right) \Big|_0^1 = \frac{3}{10} + \frac{1}{2} = \frac{8}{10} = \frac{4}{5}$.