Exam 4 **Name**______________________________

Math 231-01 **Calculus III** **November 30, 2018**

Guidelines

- **Calculators are not allowed.**
- Read the questions carefully. You have 65 minutes; use your time wisely.
- You may leave your answers in symbolic form, like $\sqrt{3}$ or $\ln(2)$, unless they simplify further like $\sqrt{9} = 3$ or $\cos(3\pi/4) = -\sqrt{2}/2$.
- Put a box around your final answers when relevant.
- Show all steps in your solutions and make your reasoning clear. Answers with no explanation will not receive full credit, even when correct.
- Use the space provided. If necessary, write "see other side" and continue working on the back of the same page.
- $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$ and $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$
- $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho \cos\varphi$, and $dV = \rho^2 \sin\varphi d\rho\, d\varphi\, d\theta$
- **Green's Theorem** Let C be a closed bounded curve that bounds a region R in the plane and let $\mathbf{F}(x,y)) = f\,\mathbf{i} + g\,\mathbf{j}$ be a vector field then

  1. Circulation: $\displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.$
  2. Outward Flux Integral: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = \oint_C f\,dy - g\,dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA$

- **Stokes' Theorem** Let S be an oriented surface in $\mathbb{R}^3$ with a piecewise-smooth closed boundary C whose orientation is consistent with that of S. Assume that $\mathrm{F} = \langle f, g, h\rangle$ is a vector field whose components have continuous first partial derivatives on S. Then

  $$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

  when n is the unit vector normal to S determined by the orientation of S.

Question	Points	Score
1	8	
2	10	
3	10	
4	10	
5	10	
6	10	
7	12	
8	10	
9	10	
10	10	
Total:	100	

1. (8 points) Complete test corrections.

2. (10 points) Evaluate the line integral $\int_C (x^2 + y^2 + z^2)\, ds$ where C is the helix $\mathbf{r}(t) = \langle t, \cos(2t), \sin(2t)\rangle$ for $0 \le t \le 2\pi$.

Solution:

Given $\mathbf{r}(t) = \langle t, \cos(2t), \sin(2t)\rangle$ then

$ds = |\mathbf{r}'|dt = |\langle 1, -2\sin(2t), 2\cos(2t)\rangle|dt = \sqrt{1 + 4\sin^2(2t) + 4\cos^2(2t)}\, dt = \sqrt{5}\, dt$

$$\begin{aligned}\int_C (x^2 + y^2 + z^2)\, ds &= \int_0^{2\pi} \left(t^2 + \cos^2(2t) + \sin^2(2t)\right)\sqrt{5}\, dt \\ &= \sqrt{5}\int_0^{2\pi} \left(t^2 + 1\right)\, dt \\ &= \sqrt{5}\left(\frac{t^3}{3}\bigg|_0^{2\pi} + t\bigg|_0^{2\pi}\right) = \sqrt{5}\left(\frac{8\pi^3}{3} + 2\pi\right)\end{aligned}$$

3. (10 points) Compute the divergence and curl of the vector field $\mathbf{F} = \langle xy^2z^2, x^2yz^2, x^2y^2z\rangle$. State whether the field is source free or irrotational.

Solution:

The divergence is $\nabla \cdot \mathbf{F} = \dfrac{\partial(xy^2z^2)}{\partial x} + \dfrac{\partial(x^2yz^2)}{\partial y} + \dfrac{\partial(x^2y^2z)}{\partial z} = y^2z^2 + x^2z^2 + x^2y^2.$

The curl is $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy^2z^2 & x^2yz^2 & x^2y^2z \end{vmatrix} = \langle 0, 0, 0\rangle$

The field is not source free $\nabla \cdot \mathbf{F} \neq 0$; however, it is irrotational becauase $\nabla \times \mathbf{F} = \mathbf{0}$.

4. (10 points) Evaluate the surface integral $\iint_S (x+y+z)\,dS$ where S is the parallelogram with parametric equations $x = u+v$, $y = u-v$, $z = 1+2u+v$ for $0 \le u \le 2$, $0 \le v \le 1$. Set up, but do not evaluate.

Solution:

$\mathbf{r}(u,v) = \langle u+v, u-v, 1+2u+v\rangle$

First $dS = |\mathbf{r}_u \times \mathbf{r}_v| dA$

So $\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = \langle 1+2, -(1-2), -1-1\rangle = \langle 3, 1, -2\rangle$

Thus $dS = |\mathbf{r}_u \times \mathbf{r}_v| dA = \sqrt{9+1+4}\,dA = \sqrt{14}\,dA$

$$\begin{aligned}\iint_S (x+y+z)\,dS &= \int_0^1 \int_0^2 \sqrt{14}\,(u+v+u-v+1+2u+v)\;du\,dv \\ &= \sqrt{14} \int_0^1 \int_0^2 (4u+v+1)\;du\,dv\end{aligned}$$

5. (10 points) Evaluate the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ using Stokes' Theorem where $\mathbf{F} = \langle 1, x+yz, xy-\sqrt{z}\rangle$; C is the boundary of the plane $x+y+z=1$ in the first octant. Assume that C has counterclockwise orientation. Set up but do not evaluate.

Solution:

Evaluate $\iint_S (\nabla \times \mathbf{F})\cdot \mathbf{n}\,dS$

First the curl is $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 1 & x+yz & xy-\sqrt{z} \end{vmatrix} = \langle x-y, -y, 1\rangle$

Second $\mathbf{n}\,dS = \dfrac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}|\mathbf{r}_x \times \mathbf{r}_y| dA = \mathbf{r}_x \times \mathbf{r}_y\,dA$

$\mathbf{r}(x,y) = \langle x, y, 1-x-y\rangle$ for $0 \le x \le 1$, $0 \le y \le 1-x$

So $\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 0+1, -(-1-0), 1-0\rangle = \langle 1, 1, 1\rangle$

Thus $(\nabla \times \mathbf{F}) \cdot \mathbf{r}_x \times \mathbf{r}_y \, dA = (x - y - y + 1) \, dA = (x - 2y + 1) \, dA$

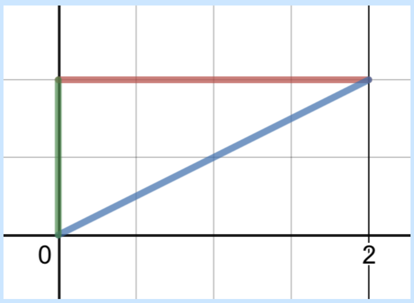
$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = \int_0^1 \int_0^{1-x} (x - 2y + 1) \; dy \, dx$$

6. (10 points) Evaluate $\oint_C (x^2+y^2)\,dx+(x^2-y^2)\,dy$, where C boundary of the triangle with vertices $(0,0)$, $(2,1)$ and $(0,1)$.

Solution:

Use Green's Theorem (Circulation)
$\mathbf{F} = \langle f, g\rangle = \langle x^2+y^2, x^2-y^2\rangle$

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = 2x - 2y$$



The lines of the triangle are
$x = 0,\ y = 1,\ y = x/2$

$$\begin{aligned}\oint_C (x^2+y^2)\,dx + (x^2-y^2)\,dy &= \iint \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\,dA \\ &= \int_0^2 \int_{x/2}^1 (2x-2y)\ dy\,dx \\ &= \int_0^2 (2xy - y^2)\Big|_{x/2}^1\ dx = \int_0^2 \left(2x - 1 - x^2 + \frac{x^2}{2}\right)\,dx \\ &= \left(x^2 - x - \frac{x^3}{3}\right)\Big|_0^2 = 4 - 2 - \frac{8}{3} = -\frac{2}{3}\end{aligned}$$

7. Given $\mathbf{F} = \langle 2xy^3z^2 + 2y^2, 3x^2y^2z^2 + 4xy + z, 2x^2y^3z + y\rangle$ and C is $\mathbf{r}(t) = \langle 1-2t, t+1, t^2\rangle$ for $0 \le t \le 1$.

   a. (6 points) Find a function φ such that $\mathbf{F} = \nabla\varphi$

Solution:
$\mathbf{F} = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z\rangle$

To find $\varphi(x,y,z)$ start by integrating each partial derivative of $\nabla\varphi$ with respect to the variable list.
$\int \varphi_x\,dx = \int (2xy^3z^2 + 2y^2)\,dx = x^2y^3z^2 + 2xy^2$
$\int \varphi_y\,dy = \int (3x^2y^2z^2 + 4xy + z)\,dy = x^2y^3z^2 + 2xy^2 + yz$

$$\int \varphi_z \, dz = \int (2x^2y^3z + y) \, dz = x^2y^3z^2 + yz$$

Combining the three antiderivative we get

$$\varphi(x, y, z) = x^2y^3z^2 + 2xy^2 + yz + C.$$

(Each element in the three anitderivatives is added only once.)

b. (6 points) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Solution:

Since **F** is conservative, it is path independent

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A)$$

where B is the point associated with $\mathbf{r}(1) = \langle 1 - 2, 1 + 1, 1^2 \rangle = \langle -1, 2, 1 \rangle$ so B is the point $(-1, 2, 1)$ and A is the point associated with $\mathbf{r}(0) = \langle 1, 1, 0 \rangle$ so A is the point $(1, 1, 0)$.

Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(-1, 2, 1) - \varphi(1, 1, 0) = 8 - 8 + 2 - 0 - 2 - 0 = 0$$

8. (10 points) Use Stokes' Theorem to evaluate the surface integral $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ where $\mathbf{F} = \langle x^2 \sin z, y^2, xy \rangle$; S the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upward. Set up, but do not evaluate.

Solution:

Evaluate $\oint \mathbf{F} \cdot d\mathbf{r}$

Note: when $z = 0$ we get $0 = 1 - x^2 - y^2 \iff 1 = x^2 + y^2 \implies r = 1$ so C is a circle of radius 1 centered at the origin. So C is $\mathbf{r}(\theta) = \langle \cos\theta, \sin\theta, 0 \rangle$ for $0 \le \theta \le 2\pi$. This circle is traversed counterclockwise, assuming the the normal vector for the surface was upward.

Now $d\mathbf{r} = \langle -\sin\theta, \cos\theta, 0 \rangle$ and
$\mathbf{F} = \langle x^2 \sin z, y^2, xy \rangle = \left\langle 0, \sin^2\theta, \cos\theta \sin\theta \right\rangle$

$$\oint \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \sin^2\theta \cos\theta \, d\theta = \frac{1}{3} \sin^3\theta \Big|_0^{2\pi} = 0$$

9. (10 points) Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ for $\mathbf{F} = \langle -x, -y, z^3 \rangle$; S is the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 3$. Set up, but do not evaluate.

Solution:

First $\mathbf{n}\, dS = \dfrac{\mathbf{r}_r \times \mathbf{r}_\theta}{|\mathbf{r}_r \times \mathbf{r}_\theta|} |\mathbf{r}_r \times \mathbf{r}_\theta| dA = \mathbf{r}_r \times \mathbf{r}_\theta \, dA$

$\mathbf{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r \rangle$ for $1 \le r \le 3$, $0 \le \theta \le 2\pi$

So $\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r \rangle$

Second, $\mathbf{F} = \langle -x, -y, z^3 \rangle = \langle -r\cos\theta, -r\sin\theta, r^3 \rangle$

Thus $\mathbf{F} \cdot \mathbf{r}_r \times \mathbf{r}_\theta \, dA = (r^2 \cos^2\theta + r^2 \sin^2\theta + r^4)\, dA = (r^2 + r^4)\, dA$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \int_0^{2\pi} \int_1^3 (r^2 + r^4)\, dr\, d\theta$$

10. (10 points) Find the work done by $\mathbf{F} = \langle xy, y, -yz \rangle$ over the curve C: $\mathbf{r}(t) = \langle t, t^2, t \rangle$ as t increases from $t = 0$ to $t = 1$.

Solution:

Work is $\displaystyle\int \mathbf{F} \cdot d\mathbf{r}$

Now $\mathbf{F} = \langle xy, y, -yz \rangle = \langle t(t^2), t^2, -t^2(t) \rangle = \langle t^3, t^2, -t^3 \rangle$

and $d\mathbf{r} = \langle 1, 2t, 1 \rangle \, dt$.

$\mathbf{F} \cdot d\mathbf{r} = (t^3 + 2t^3 - t^3) \, dt = 2t^3 \, dt$.

Thus $\displaystyle\int \mathbf{F} \cdot d\mathbf{r} = \int_0^1 2t^3 \, dt = \frac{1}{2} t^4 \Big|_0^1 = \frac{1}{2}$.