Exam 3 **Name**________________________________
Math 231-03 **Calculus III** **November 2, 2018**

Guidelines

- **Calculators are not allowed.**
- Read the questions carefully. You have 65 minutes; use your time wisely.
- You may leave your answers in symbolic form, like $\sqrt{3}$ or $\ln(2)$, unless they simplify further like $\sqrt{9} = 3$ or $\cos(3\pi/4) = -\sqrt{2}/2$.
- Put a box around your final answers when relevant.
- Show all steps in your solutions and make your reasoning clear. Answers with no explanation will not receive full credit, even when correct.
- Use the space provided. If necessary, write "see other side" and continue working on the back of the same page.
- $\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}|| \cos\theta$ and $||\vec{u} \times \vec{v}|| = ||\vec{u}||||\vec{v}|| \sin\theta$
- $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho \cos\varphi$, and $dV = \rho^2 \sin\varphi d\rho\, d\varphi\, d\theta$

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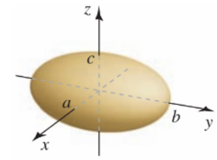
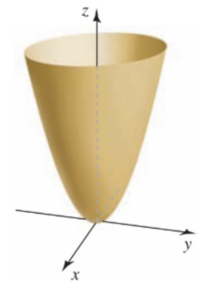
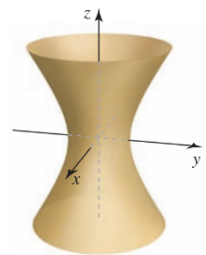
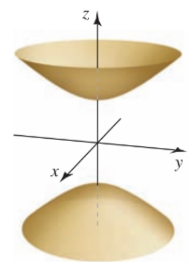
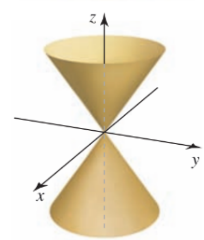
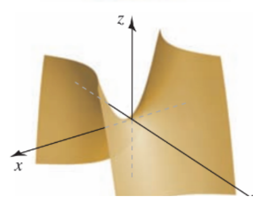
1. (8 points) To be completed once exams are graded and returned. Please correct any problem with points deducted. All corrections should be completed neatly on a separate sheet of paper. Once you have finished your corrections, take your exam and corrections to the Office of Student Learning (OSL), and a tutor will check your answers and sign below. The checked solutions should be given to your instructor.

   Signature: ____________________________
   Print Name: ____________________________
   Date: ______________

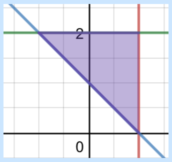
Table 12.1

Name	Standard Equation	Features	Graph
Ellipsoid	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$	All traces are ellipses.	
Elliptic paraboloid	$z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$	Traces with $z = z_0 > 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are parabolas.	
Hyperboloid of one sheet	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$	Traces with $z = z_0$ are ellipses for all z_0. Traces with $x = x_0$ or $y = y_0$ are hyperbolas.	
Hyperboloid of two sheets	$-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$	Traces with $z = z_0$ with $\lvert z_0 \rvert > \lvert c \rvert$ are ellipses. Traces with $x = x_0$ and $y = y_0$ are hyperbolas.	
Elliptic cone	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$	Traces with $z = z_0 \neq 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are hyperbolas or intersecting lines.	
Hyperbolic paraboloid	$z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$	Traces with $z = z_0 \neq 0$ are hyperbolas. Traces with $x = x_0$ or $y = y_0$ are parabolas.	

Question	Points	Score
1	8	
2	10	
3	10	
4	10	
5	14	
6	10	
7	10	
8	14	
9	14	
Total:	100	

2. (10 points) For $\int_0^2 \int_{1-y}^1 f(x,y)\,dx\,dy$, sketch the region R and re-write as a double integral with the order of integration reversed.

Solution:



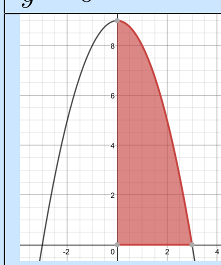
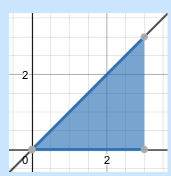
$$\int_{-1}^{1} \int_{1-x}^{2} f(x,y)\,dy\,dx$$

3. (10 points) For $\int_0^3 \int_0^{9-x^2} \int_0^x f(x,y,z)\,dy\,dz\,dx$, sketch the region D and re-write as an iterated integral in the order $dz\,dx\,dy$

Solution:

From the given integral, we know that $0 \le x \le 3$, $0 \le z \le 9 - x^2$ and $0 \le y \le x$. From this, we can see that this is the wedge of parabolic cylinder $z = 9 - x^2$ between the planes $z = 0$, $x = 0$, $y = 0$, and $y = x$.

Looking at the cross sections below, we see the bounds for x,y, and z.

$y = 0$	$z = 0$
	

Thus, we see that $0 \le z \le 9 - x^2$, $y \le x \le 3$, and $0 \le y \le 3$. So,

$$\int_0^3 \int_0^{9-x^2} \int_0^x f(x,y,z)\,dy\,dz\,dx = \int_0^3 \int_x^3 \int_0^{9-x^2} f(x,y,z)\,dz\,dx\,dy$$

4. (10 points) Rewrite the integral $\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$ in spherical coordinates. Do no evaluate.

Solution:

We see that $-\sqrt{4-x^2-y^2} \leq z \leq \sqrt{4-x^2-y^2}$, so we're dealing with a sphere of radius 2 ($\rho = 2$). Similarly, $0 \leq y \leq \sqrt{4-x^2}$ and $-2 \leq x \leq 2$ so a hemisphere from the origin to 4 along the $+y$-axis. Thus, $0 \leq \varphi \leq \pi$ and $0 \leq \theta \leq \pi$.

Substituting variables into the integrand, we get $\sqrt{x^2+y^2+z^2} = \rho$ and $y^2 = \rho^2 \sin^2 \varphi \sin^2 \theta$. Plugging in new bounds and variables, and simplifying we get that:

$$\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2 \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx = \int_{0}^{\pi} \int_{0}^{\pi} \int_{0}^{2} \rho^5 \sin^3 \varphi \sin^2 \theta d\rho d\varphi d\theta$$

5. (14 points) Find the average value of $z = \sqrt{16 - x^2 - y^2}$ over the disk in the xy-plane centered at the origin with radius 4.

Solution:

When $z = 0$ then the disk is $x^2 + y^2 = 16$ The average value is

$$
\begin{aligned}
\overline{f} &= \frac{1}{\text{area of disk}} \iint_R \sqrt{16 - x^2 - y^2}\, dA \\
&= \frac{1}{16\pi} \int_{-4}^{4} \int_{\sqrt{16-x^2}}^{\sqrt{16-x^2}} \sqrt{16 - x^2 - y^2}\, dydx \\
&= \frac{1}{16\pi} \int_0^{2\pi} \int_0^4 \sqrt{16 - r^2}\, rdrd\theta \\
&= \frac{1}{16\pi} \int_0^{2\pi} -\frac{1}{2} \cdot \frac{2}{3} (16 - r^2)^{3/2} \Big|_0^4 \, d\theta \\
&= \frac{1}{16\pi} \int_0^{2\pi} -\frac{1}{3} (0 - 16^{3/2})\, rdrd\theta \\
&= \frac{1}{16\pi} \cdot \frac{64}{3} \cdot 2\pi \\
&= \frac{8}{3}
\end{aligned}
$$

6. (10 points) Convert $\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3r\, dzdrd\theta$ to rectangular coordinates with order of integration $dzdxdy$. Do not evaluate.

Solution:

$0 \le \theta \le 2\pi$, $0 \le r \le \sqrt{2}$ and $r \le z \le \sqrt{4 - r^2}$

So converting $\sqrt{x^2 + y^2} \le z \le \sqrt{4 - x^2 - y^2}$ this looks like a snow cone. These two surfaces meet in a circle centered at the origin with radius $\sqrt{2}$.

$$
\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3r\, dzdrd\theta = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} 3\, dzdxdy
$$

7. (10 points) Find the volume of the solid that is enclosed by the planes $x = 2$, $y = 0$, $z = 0$, $y = x$ and the parabolic cylinders $z = x^2$. Set up the triple integral but do not evaluate.

Solution:

$$V = \int_0^2 \int_0^x \int_0^{x^2} dz\, dy\, dx$$

8. (14 points) Use cylindrical coordinates to find the volume of the solid bounded by the plane $z = \sqrt{29}$ and the hyperboloid $z = \sqrt{4 + x^2 + y^2}$.

Solution:

These two surface meet when $\sqrt{29} = \sqrt{4 + x^2 + y^2} \implies 29 = 4 + x^2 + y^2 \implies x^2 + y^2 = 25$ so the intersect in a circle centered at the origin with radius 5. Thus $r = 5$.

Now in cylindrical coordinates $z = z$ so $z = \sqrt{4 + x^2 + y^2} = \sqrt{4 + r^2}$
Thus,

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^5 \int_{\sqrt{4+r^2}}^{\sqrt{29}} r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^1 \left(\sqrt{29} - \sqrt{4 + r^2}\right) r\, dr\, d\theta \\
&= \int_0^{2\pi} \left(\sqrt{29}\frac{r^2}{2} - \frac{1}{2} \cdot \frac{2}{3}(4 + r^2)^{3/2}\right)\Bigg|_0^5 d\theta \\
&= \int_0^{2\pi} \left(\sqrt{29}\left(\frac{25}{2}\right) - \frac{1}{3}\left(29\sqrt{29} - 8\right)\right) d\theta \\
&= 2\pi\left(\sqrt{29}\left(\frac{17}{6}\right) - \frac{8}{3}\right)
\end{aligned}$$

9. (14 points) Use a transformation and evaluate $\iint_R e^{xy} dA$ where R is the region bounded by the hyperbolas $xy = 1$ and $xy = 4$, and the lines $y/x = 1$ and $y/x = 3$ in the first quadrant.

Solution:

Let $u = xy$, from the problem $1 \le xy \le 4 \implies 1 \le u \le 4$ and let $v = y/x$ again from above $1 \le y/x \le 3 \implies 1 \le v \le 3$.

Solving for x and y from the substitution, we see that $y = u/x$ plugging this into $v = y/x = \dfrac{u}{x^2}$ so $x^2 = \dfrac{u}{v} \implies x = \dfrac{u^{1/2}}{v^{1/2}} = u^{1/2}v^{-1/2}$ and $y = \dfrac{u}{x} = \dfrac{u}{u^{1/2}v^{-1/2}} = u^{1/2}v^{1/2}$

For the Jacobian,

$$J = \begin{vmatrix} \frac{1}{2}u^{-1/2}v^{-1/2} & -\frac{1}{2}u^{1/2}v^{-3/2} \\ \frac{1}{2}u^{-1/2}v^{1/2} & \frac{1}{2}u^{1/2}v^{-1/2} \end{vmatrix} = \frac{1}{4}v^{-1} + \frac{1}{4}v^{-1} = \frac{1}{2v}$$

$$\begin{aligned} \iint_R e^{xy}\, dA &= \int_1^3 \int_1^4 e^u \left| -\frac{1}{2v} \right| du\, dv \\ &= \int_1^3 \frac{1}{2v} e^u \Big|_1^4 dv \\ &= \int_1^3 \frac{1}{2v} \left(e^4 - e\right) dv \\ &= \frac{1}{2}\left(e^4 - e\right) \ln|v| \Big|_1^3 \\ &= \frac{\ln 3}{2}\left(e^4 - e\right) \end{aligned}$$