1. Find the equation of the plane passing through the points $(1, 2, 1)$, $(-1, 3, 2)$, and $(0, -1, 5)$.

Solution:

Label the points $P(1, 2, 1)$, $Q(-1, 3, 2)$, and $R(0, -1, 5)$. To determine a plane, we need a vector normal to the plane. The vector $\mathbf{n} = \overline{PQ} \times \overline{PR}$ will be orthogonal to the plane containing P, Q, and R.

$\overline{PQ} = \langle -2, 1, 1\rangle$ and $\overline{PR} = \langle -1, -3, 4\rangle$

$$\mathbf{n} = \overline{PQ} \times \overline{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 1 \\ -1 & -3 & 4 \end{vmatrix} = \langle 7, 7, 7\rangle$$

The plane is $7(x - 1) + 7(y - 2) + 7(z - 1) = 0$ or $7x + 7y + 7z = 28$ or $x + y + z = 4$.

2. Determine whether the planes $3x + 2y - 3z = 10$ and $-6x - 10y + z = 10$ are parallel, orthogonal or neither. If neither, what is the angle between the two planes?

Solution:

The normal vector for the first plane is $\mathbf{n}_1 = \langle 3, 2, -3\rangle$ and for the second plane is $\mathbf{n}_2 = \langle -6, -10, 1\rangle$, respectively.

Since the normal vectors are not parallel, $\mathbf{n}_2 \neq c\,\mathbf{n}_2$ for any real number c, thus the planes are NOT parallel.

The planes are not orthogonal since the normal vectors are not orthogonal, $\mathbf{n}_1 \times \mathbf{n}_2 = -41 \neq 0$.

The angle between the planes is $\theta = \cos^{-1}\left(\dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right) \cos^{-1} = \left(\dfrac{-41}{\sqrt{22}\sqrt{137}}\right)$

Note $\mathbf{n}_1 \cdot \mathbf{n}_2 = 3(-6) + 2(-10) - 3(1) = -41$, $|\mathbf{n}_1| = \sqrt{9 + 4 + 9} = \sqrt{22}$
and $\mathbf{n}_2| = \sqrt{36 + 100 + 1} = \sqrt{137}$.

3. Find an equation of the line of intersection of the planes $Q : 2x - y + 3z - 1 = 0$ and $R : -x + 3y + z - 4 = 0$

Solution:

The direction vector for the line $\mathbf{v}$ is the vector that is orthogonal to the normal vectors $\mathbf{n}_Q = \langle 2, -1, 3\rangle$ and $\mathbf{n}_R = \langle -1, 3, 1\rangle$.

$$\mathbf{v} = \mathbf{n}_Q \times \mathbf{n}_R = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ -1 & 3 & 1 \end{vmatrix} = \langle -10, -5, 5 \rangle$$

Find a point on the line: Let $x = 0$ then $-y + 3z = 1$ and $3y + z = 4$. From the first one, $y = 3z - 1$, plugging this into the second $3(3z - 1) + z = 4 \iff 10z = 7 \iff z = \frac{7}{10}$ and $y = 3\frac{7}{10} - 1 = \frac{11}{10}$, thus a point on the line is $\left(0, \frac{11}{10}, \frac{7}{10}\right)$.

The equation of the line of intersection

$$\mathbf{r}(t) = \left\langle 0, \frac{11}{10}, \frac{7}{10} \right\rangle + t \langle -10, -5, 5 \rangle \quad t \epsilon \mathbb{R}$$

OR $\mathbf{r}(t) = \left\langle -10t, \frac{11}{10} - 5t, \frac{7}{10} + 5t \right\rangle$ for $t \epsilon \mathbb{R}$.
OR $x = -10t$, $y = \frac{11}{10} - 5t$, $z = \frac{7}{10} + 5t$ for $t \epsilon \mathbb{R}$.

4. Consider the cylinder $x = z^2 - 4$ in $\mathbb{R}^3$. Identify the coordinate axis to which the cylinder is parallel. Sketch the cylinder.

Solution:

5. Identify and briefly describe the surface $x^2 + y^2 + z^2 + 2x - 4y - 16 = 0$.

Solution:
$x^2 + 2x + 1 - 1 + y^2 - 4y + 4 - 4 + z^2 - 16 = 0 \iff (x + 1)^2 + (y - 2)^2 + z^2 = 21$

The quadric surface is a sphere; centered at $(-1, 2, 0)$ with radius $\sqrt{21}$.

6. Identify and briefly describe the surface $y = 4x^2 + z^2$.

Solution:
The quadric surface is an elliptical paraboloid. It looks like an elliptical bowl on its side opening along the positive y-axis.

7. Find $\frac{\partial z}{\partial x}$ for $xe^{yz} + ye^{xz} + ze^{xy} = 5$, assuming that $z = f(x, y)$

Solution:
For the given equation, $F(x, y, z) = xe^{yz} + ye^{xz} + ze^{xy} - 5 = 0$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{e^{yz} + yze^{xz} + yze^{xy}}{xye^{yz} + xye^{xz} + e^{xy}}.$$

8. Find the linear approximation to the function, $f(x, y) = \sqrt{x^2 + y^2}$ at the point $P(3, -4)$. Use it to estimate $f(3.06, -3.92)$.

Solution:

Find the tangent plane to $z - \sqrt{x^2 + y^2} = 0$ at the point $P(3, -4, f(-3, 4) = 5)$

The normal vector for the tangent plane is $\nabla F = \left\langle \frac{-2x}{\sqrt{x^2+y^2}}, \frac{-2y}{\sqrt{x^2+y^2}}, 1 \right\rangle$, so $\mathbf{n} = \nabla F(3, -4, 5) = \left\langle \frac{-3}{5}, \frac{4}{5}, 1 \right\rangle$

So the tangent plane is $\frac{-3}{5}(x - 3) + \frac{4}{5}(y + 4) + z - 5 = 0$

The linear approximation

$$L(x, y) = \frac{3}{5}(x - 3) - \frac{4}{5}(y + 4) + 5$$

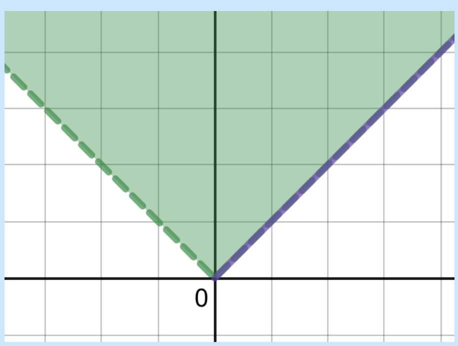
And

$$f(3.06, -3.92) = \frac{3}{5}\left(\frac{6}{100}\right) - \frac{4}{5}\left(-\frac{8}{100}\right) + 5 = 5 + \frac{50}{500} = 5.1$$

9. Sketch the domain of the function $f(x, y) = \sqrt{y - x}\ln(y + x)$ in the xy-plane.

Solution:

For the domain, $y - x \geq 0$ AND $y + x > 0$, thus $y \geq x$ AND $y > -x$.



10. Given the function $f(x, y, z) = e^{xy^2z^3}$

a. Find $\frac{\partial f}{\partial x}$

Solution:

$\frac{\partial f}{\partial x} = y^2 z^3 e^{xy^2z^3}$

b. Find $\dfrac{\partial^2 f}{\partial x^2}$

Solution:
$\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = y^4 z^6 e^{xy^2z^3}$

c. Find $\dfrac{\partial^2 f}{\partial x \partial y}$

Solution:
$\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = 2yz^3 e^{xy^2z^3} + 2xy^3z^6 e^{xy^2z^3} = \left(2yz^3 + 2xy^3z^6\right) e^{xy^2z^3}$

d. Find $\dfrac{\partial^3 f}{\partial z \partial y \partial x}$

Solution:

$$\begin{aligned}
\frac{\partial^3 f}{\partial z \partial y \partial x} &= \frac{\partial^3 f}{\partial x \partial y \partial z} \\
&= \frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial x \partial y}\right) \\
&= \left(2y(3z^2) + 2xy^3(6z^5)\right) e^{xy^2z^3} + \left(2yz^3 + 2xy^3z^6\right)(3xy^2z^2)e^{xy^2z^3} \\
&= \left(6yz^2 + 18xy^3z^5 + 6x^2y^5z^8\right) e^{xy^2z^3}
\end{aligned}$$

11. Find the directional derivative to the surface given by the function $f(x, y) - 7 + 10x\sqrt{y}$ at the point $P(5, 16)$ in the direction of the vector $\vec{v} = \langle -4, 3 \rangle$.

Solution:
$\nabla f = \langle f_x, f_y \rangle = \left\langle 10\sqrt{y}, \dfrac{10x}{2\sqrt{y}} \right\rangle$, so $\nabla f(5, 16) = \left\langle 10\sqrt{16}, \dfrac{5(5)}{\sqrt{16}} \right\rangle = \left\langle 40, \dfrac{25}{4} \right\rangle$.

Unit vector for directions $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{\langle -4, 3 \rangle}{\sqrt{16+9}} = \left\langle -\dfrac{4}{5}, \dfrac{3}{5} \right\rangle$.

So the directional derivative is

$$D_{\mathbf{u}} f(-4, 3) = \left\langle 40, \frac{25}{4} \right\rangle \cdot \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle = 40\left(\frac{-4}{5}\right) + \frac{25}{4}\left(\frac{3}{5}\right) = -32 + \frac{15}{4} = -\frac{113}{5}$$

12. Find an equation of the plane tangent to the surface given by the function $f(x, y) = x^3 - x^2 y$ at the point $P(2, 1)$.

Solution:
Let $F(x, y, z) = z - x^3 + x^2 y = 0$ so $\nabla F = \langle F_x, F_y, F_z \rangle = \left\langle -3x^2 + 2xy, x^2, 1 \right\rangle$

The point on the plane is $(2, 1, f(2, 1) = 2^3 - 2^2 \cdot 1 = 4) = (2, 1, 4)$.

And $\nabla F(2,1,4) = \langle -8, 4, 1 \rangle$.

Thus the tangent plane to the surface at P is

$$\begin{aligned} -8(x-2) + 4(y-1) + z - 4 &= 0 \\ z &= 8(x-2) - 4(y-1) + 4 \\ z &= 8x - 4y - 8 \end{aligned}$$

13. For $f(x,y) = x^2 - y^2$, find a line in the x direction tangent to the surface defined by f at $(1,2)$.

Solution:

The direction vector for the tangent line is $\mathbf{v} = \langle 1, 0, D_u f(1,2) \rangle$. Note the first two components are known because we are going in the x direction to the surface. The direction from the point $P(1,2)$ is $\mathbf{u} = \langle 1, 0 \rangle = \mathbf{i}$.

$\nabla f = \langle f_x, f_y \rangle = \langle 2x, -2y \rangle$, so $\nabla f(1,2) = \langle 2, -4 \rangle$

$D_u f(1,2) = \langle 2, -4 \rangle \cdot \langle 1, 0 \rangle = 2$

Thus direction vector for the tangent line is $\mathbf{v} = \langle 1, 0, D_u f(1,2) \rangle = \langle 1, 0, 2 \rangle$.

The point on the tangent line is $P(1, 2, f(1,2) = 1^2 - 2^2 = -3) = P(1,2,-3)$

Tangent line in the x direction is

$$\begin{aligned} \mathbf{r}(t) &= \langle 1, 2, -3 \rangle + t \langle 1, 0, 2 \rangle \quad t \epsilon \mathbb{R} \\ \text{OR} \quad \mathbf{r}(t) &= \langle 1 + t, 2, -3 + 2t \rangle \end{aligned}$$

14. Suppose that the temperature w (in degrees Celsius) at the point (x,y) is given by $w = f(x,y) = 5 + 0.002x^2 + 0.003y^2$. In what direction should the grasshopper hop from the point $(10,20)$ to get warmer as quickly as possible? What is the rate of change of the temperature in this direction?

Solution:

$\nabla w = \langle f_x, f_y \rangle = \langle 0.004x, 0.006y \rangle \implies \nabla w(10,20) = \langle 0.04, 0.12 \rangle$

The direction to get warm as quickly as possible (the direction of the maximum directional derivative, $D_{\mathbf{u}} f = |\nabla w| \cos\theta$) is in the direction of the gradient ($\theta = 0$), so direction is $\nabla w(10,20) = \langle 0.04, 0.12 \rangle$ OR $\dfrac{\langle 0.04, 0.12 \rangle}{|\langle 0.04, 0.12 \rangle|} = \dfrac{1}{0.126} \langle 0.04.0.12 \rangle$.

The rate of change in this direction is $|\nabla w(10,20)| = \sqrt{(0.04)^2 + (0.12)^2} = 0.126$

15. Find the directional derivative to the function $f(x, y, z) = 2z\sqrt{xy}$ at the point $(2, 2, 3)$ in the direction of the vector $\mathbf{v} = \langle 1, 1, 1 \rangle$.

Solution:

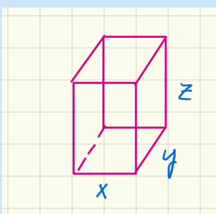
$$\nabla f = \langle f_x, f_y, f_z \rangle = \left\langle \frac{yz}{\sqrt{xy}}, \frac{xz}{\sqrt{xy}}, 2\sqrt{xy} \right\rangle \implies \nabla f(2, 2, 3) = \langle 3, 3, 4 \rangle.$$

The unit vector for direction is $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{\langle 1, 1, 1 \rangle}{\sqrt{1+1+1}} = \dfrac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$.

Thus $D_{\mathbf{u}} f(2, 2, 3) = \langle 3, 3, 4 \rangle \cdot \left\langle \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}} \right\rangle = \dfrac{3}{\sqrt{3}} + \dfrac{3}{\sqrt{3}} + \dfrac{4}{\sqrt{3}} = \dfrac{10}{\sqrt{3}} = \dfrac{10\sqrt{3}}{3}$

16. Use the method of Lagrange mulitipliers to find the dimensions of a rectangular box (with a lid) with largest volume, if the total surface area is 96 cm^2.

Solution:



The volume is $V = xyz$ where $x > 0, y > 0, z > 0$
Constraint: Surface area is 96, $S = 2xy + 2yz + 2xz = 96$ OR
$g(x, y, z) = xy + yz + xz - 48 = 0$

The gradient vectors are $\nabla V = \langle V_x, V_y, V_z \rangle = \langle yz, xz, xy \rangle$ and $\nabla g = \langle y + z, x + z, x + y \rangle$.

Solve $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 0$

$$yz = \lambda(y + z) \quad (1)$$
$$xz = \lambda(x + z) \quad (2)$$
$$xy = \lambda(x + y) \quad (3)$$
$$48 = xy + yz + xz \quad (4)$$

From equations (1), (2), and (3), $xyz = \lambda(xy + xz) + \lambda(xy + yz) + \lambda(xz + yz)$, thus

$$\begin{aligned}\lambda(xy + xz) = \lambda(xy + yz) &\iff \lambda(xy + xz) - \lambda(xy + yz) = 0 \\ &\iff \lambda(xy + xz - xy - yz) = \lambda z(x - y) = 0\end{aligned}$$

Since $z \neq 0$, the solution is $x = y$

$$\begin{aligned}\lambda(xy + xz) = \lambda(xz + yz) &\iff \lambda(xy + xz) - \lambda(xz + yz) = 0 \\ &\iff \lambda(xy + xz - xz - yz) = \lambda y(x - z) = 0\end{aligned}$$

Since $y \neq 0$, the solution is $x = z$

Thus $x = y = z$,
$g(x, x, x) = x^2 + x^2 + x^2 - 48 = 0 \iff 3x^2 = 48 \iff x^2 = 16 \iff x = 4$ (note $x \neq -4$)
So the dimensions of the box with the largest volume is $x = y = z = 4$.

17. Find $\frac{\partial z}{\partial t}$ for $z = sin(x^2y)$, $x = \frac{s}{t}$, and $y = t^2e^{st}$.

Solution:
$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} = 2xy\cos(x^2y) \cdot \left(-\frac{s}{t^2}\right) + x^2\cos(x^2y) \cdot \left(2te^{st} + st^2e^{st}\right)$

18. Find the limit or state that it does not exist:

$$\lim_{(x,y)\to(0,0)} e^{\sin(x+y-\pi/2)}$$

Solution:

$$\lim_{(x,y)\to(0,0)} e^{\sin(x+y-\pi/2)} = e^{\sin(-\pi/2)} = e^{-1}$$

19. Show the following limit does not exist:

$$\lim_{(x,y)\to(0,0)} \frac{4xy}{3x^2+y^2}$$

Solution:
Consider the path $y = mx$ through the origin, then

$$\lim_{(x,y)\to(0,0)} \frac{4xy}{3x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{4x(mx)}{3x^2+(mx)^2} = \lim_{(x,y)\to(0,0)} \frac{4mx^2}{x^2(3+m^2)} = \frac{4m}{3+m^2}$$

Thus the limit does not exist because the value of the limit changes for different values of m.

20. Find ∇f for $f(x,y,z) = xy + xz + yz + 4$.

Solution:
$\nabla f = \langle f_x, f_y, f_z\rangle = \langle y+z, x+z, x+y\rangle$

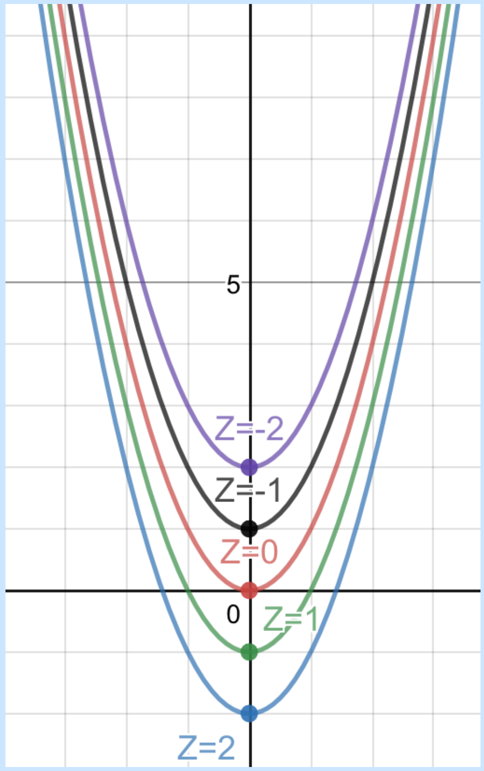
21. Find the direction in which the function $f(x,y,z) = xe^z - ye^x$ decreases most rapidly from the point $P(0,2,0)$.

Solution:
$\nabla f = \langle f_x, f_y, f_z\rangle = \langle e^z - ye^x, -e^x, xe^z\rangle$ so $\nabla f(0,2,0) = \left\langle e^0 - 2e^0, -e^0, 0e^0\right\rangle = \langle -1,-1,0\rangle$.

So the direction in which the function decreases most rapidly is opposite $\nabla f(P) = -\langle -1,-1,0\rangle = \langle 1,1,0\rangle$. (Remember the directional derivative is $D_{\mathbf{u}}f(P) = |\nabla f(P)|\cos\theta$, so the maximum rate of change is when $\theta = 0$ and it is minimized when $\theta = \pi$.)

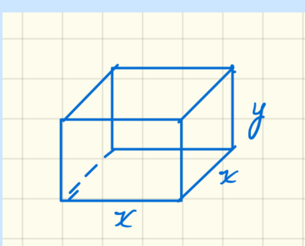
22. For the function $f(x, y) = x^2 - y$, make a sketch of several level curves. Label at least two level curves with their z-values.

Solution:



23. A rectangular box has a square base. Find the rate at which its volume is changing if its base edge is increasing at 2 cm/min and its height is decreasing at 3 cm/min at the instant when each dimension is 1 meter.

Solution:



Given: $\dfrac{dx}{dt} = 2cm/min$ and $\dfrac{dy}{dt} = -3cm/min.$
The volume is $V = x^2y$.
Now from the chain rule:

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial V}{\partial y} \cdot \frac{dy}{dt} \\ &= 2xy\frac{dx}{dt} + x^2\frac{dy}{dt} \\ \left.\frac{dV}{dt}\right|_{(100,100)} &= 2(100)(100)(2) + (100)^2(-3) \\ &= 4(100)^2 - 3(100)^2 = 100^2 = 10,000. \end{aligned}$$

24. Use the method of Lagrange Multipliers to find the minimum value of the function $f(x, y) = x^2 + y + 2z$ subject to the constrain $x^2 + 2y^2 + z^2 = 1$.

Solution:

The gradient vectors are $\nabla f = \langle f_x, f_y, f_z\rangle = \langle 2x, 1, 2\rangle$ and $\nabla g = \langle 2x, 4y, 2z\rangle$.

Solve $\nabla V = \lambda \nabla g$ and $g(x, y, z) = x^2 + 2y^2 + z^2 - 1 = 0$

$$2x = 2\lambda x \tag{5}$$
$$1 = 4\lambda y \tag{6}$$
$$2 = 2\lambda z \tag{7}$$
$$1 = x^2 + 2y^2 + z^2 \tag{8}$$

From equation (5), $2x = 2\lambda x \iff 2x - 2\lambda x = 2x(1-\lambda) = 0$ so $x = 0$ or $\lambda = 1$.

From equations (2) and (3):
$1 = 4\lambda y = \lambda z \implies 4\lambda y - \lambda z = \lambda(4y - z) = 0 \implies z = 4y$ (remember $\lambda = 1$)
With $x = 0$ and $z = 4y$, we use the constraint
(4) $g(0, y, 4y) = 0^2 + 2y^2 + (4y)^2 = 1 \iff 18y^2 = 1 \iff y^2 = \dfrac{1}{18} \iff y = \pm\sqrt{\dfrac{1}{18}} = \pm\dfrac{1}{3\sqrt{2}} = \pm\dfrac{\sqrt{2}}{6}$

So $z = 4y \implies z = 4\left(\pm\dfrac{1}{3\sqrt{2}}\right) = \pm\dfrac{2\sqrt{2}}{3}$.

Look at the function f, for the points $\left(0, \dfrac{\sqrt{2}}{6}, \dfrac{2\sqrt{2}}{3}\right)$ and $\left(0, -\dfrac{\sqrt{2}}{6}, -\dfrac{2\sqrt{2}}{3}\right)$:

$f\left(0, \dfrac{\sqrt{2}}{6}, \dfrac{2\sqrt{2}}{3}\right) = 0^2 + \dfrac{\sqrt{2}}{6} + 2\left(\dfrac{2\sqrt{2}}{3}\right) = \dfrac{\sqrt{2}}{6} + \dfrac{4\sqrt{2}}{3} = \dfrac{9\sqrt{2}}{6} = \dfrac{3\sqrt{2}}{2}$
$f\left(0, -\dfrac{\sqrt{2}}{6}, -\dfrac{2\sqrt{2}}{3}\right) = 0^2 - \dfrac{\sqrt{2}}{6} - 2\left(\dfrac{2\sqrt{2}}{3}\right) = -\dfrac{\sqrt{2}}{6} - \dfrac{4\sqrt{2}}{3} = -\dfrac{9\sqrt{2}}{6} = -\dfrac{3\sqrt{2}}{2}$, this is the MINIMUM

25. Locate and classify the critical points of the function $f(x, y) = 3xy - x^2y - xy^2$.

Solution:

CP are points where $f_x = f_y = 0$ or either f_x or f_y are undefined.

Start with $f_x = 3y - 2xy - y^2 = y(3 - 2x - y) = 0$ so either $y = 0$ or $y = 3 - 2x$
Now consider $f_y = 3x - x^2 - 2xy = 0$

if $y = 0$, then $f_y = 3x - x^2 = x(3 - x) = 0 \implies x = 0$ or $x = 3$. So we have critical points $(0, 0)$ and $(3, 0)$.

if $y = 3 - 2x$, then $f_y = 3x - x^2 - 2x(3 - 2x) = 3x^2 - 3x = 3x(x - 1) = 0 \implies x = 0$ or $x = 1$. So we have critical points $(0, 3 - 2(0) = 3) = (0, 3)$ and $(1, 3 - 2(1) = 1) = (1, 1)$.

For classification, the discriminant

$$\begin{aligned} D &= f_{xx}f_{yy} - (f_{xy})^2 \\ &= (-2y)(-2x) - (3 - 2x - 2y)^2 \\ &= 4xy - (3 - 2x - 2y)^2 \end{aligned}$$

CP	$D = 4xy - (3 - 2x - 2y)^2$	$f_{xx} = -2y$	Classification
(0,0)	$-9 < 0$		Saddle point
(3,0)	$-9 < 0$		Saddle point
(0,3)	$-9 < 0$		Saddle point
(1,1)	$3 > 0$	$-2 < 0$ CD	Local Max

For additional problems, check out the review problems for Chapter 12. Note the questions above are simply a sample of possible questions possible for the exam; it is possible that other types of questions may appear on your exam.