Exam 2 **Name** Solution

Math 231-02 **Calculus III** **March 9, 2018**

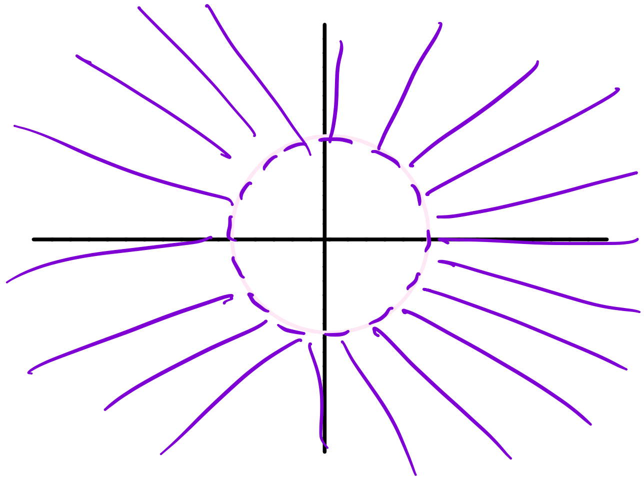
Guidelines

- **Calculators are not allowed.**
- Read the questions carefully. You have 65 minutes; use your time wisely.
- You may leave your answers in symbolic form, like $\sqrt{3}$ or $\ln(2)$, unless they simplify further like $\sqrt{9} = 3$ or $\cos(3\pi/4) = -\sqrt{2}/2$.
- Put a box around your final answers when relevant.
- Show all steps in your solutions and make your reasoning clear. Answers with no explanation will not receive full credit, even when correct.
- Use the space provided. If necessary, write "see other side" and continue working on the back of the same page.
- $\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}|| \cos\theta$ and $||\vec{u} \times \vec{v}|| = ||\vec{u}||||\vec{v}|| \sin\theta$

Question	Points	Score
1	8	
2	8	
3	12	
4	8	
5	8	
6	8	
7	8	
8	8	
9	14	
10	10	
11	8	
Total:	100	

1. (8 points) Complete test corrections.

2. (8 points) Find and sketch the domain of $f(x, y) = \dfrac{1}{\sqrt{x^2 + y^2 - 25}}$.



$x^2+y^2-25>0$

$\Rightarrow x^2+y^2>25$

Domain $\{(x,y) \mid x^2+y^2>25\}$

3. (12 points) Evaluate each limit or show it does not exist.

a. $\displaystyle\lim_{(x,y)\to(0,0)} \tan^{-1}\left(\frac{1}{x^2+y^2}\right)$ $= \displaystyle\lim_{r\to 0} \tan^{-1}\left(\frac{1}{r^2}\right) = \frac{\pi}{2}$

b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy}{3x^2+y^2}$

Let $y=mx \Rightarrow \displaystyle\lim_{(x,y)\to(0,0)} \frac{4mx^2}{3x^2+m^2x^2} = \lim_{(x,y)\to(0,0)} \frac{4m}{3+m^2} = \frac{4m}{3+m^2}$

So limit changes w/ diff values of m

$\therefore \displaystyle\lim \frac{4xy}{3x^2+y^2}$ DNE (Diff paths give diff limits)

4. (8 points) If $w = e^{xyz}$, compute $\dfrac{\partial^3 w}{\partial y^2 \partial x}$.

$w_y = xze^{xyz}$

$w_{yy} = x^2z^2e^{xyz}$

$$\boxed{w_{yyx} = \frac{\partial^3 w}{\partial y^2 \partial x} = 2xz^2e^{xyz} + x^2yz^3e^{xyz}}$$

5. (8 points) Find an equation of the tangent plane to $z = \ln(1 + xy)$ at the point $(1, 2)$.

$\vec{N} = \langle f_x, f_y, -1 \rangle = \left\langle \frac{y}{1+xy}, \frac{x}{1+xy}, -1 \right\rangle$ $\quad z = \ln(3)$

$\vec{N}_{(1,2)} = \left\langle \frac{2}{3}, \frac{1}{3}, -1 \right\rangle$

Tan plane $\left\langle \frac{2}{3}, \frac{1}{3}, -1 \right\rangle \cdot \langle x-1, y-2, z-\ln 3 \rangle = 0$

$\Rightarrow \frac{2}{3}(x-1) + \frac{1}{3}(y-2) - 1(z - \ln 3) = 0$

or $z = \ln 3 + \frac{2}{3}(x-1) + \frac{1}{3}(y-2)$

6. (8 points) Let $w = \dfrac{x - z}{y + z}$ where $x = s + t$, $y = st$, and $z = s - t$. Compute $\dfrac{\partial w}{\partial s}$.

$\frac{\partial w}{\partial x} = \frac{1}{y+z}$ $\qquad \frac{\partial w}{\partial y} = (x-z)\cdot\left((-1)(y+z)^{-2}\right)$ $\qquad \frac{\partial w}{\partial z} = \frac{-1(y+z) - (x-z)(1)}{(y+z)^2}$

$= \frac{-(x+y)}{(y+z)^2}$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= \frac{1}{y+z}(1) - \frac{x-z}{(y+z)^2}(t) - \frac{x+y}{(y+z)^2}(1)$$

7. (8 points) Find a function $f(x, y)$ satisfying $\nabla f(x, y) = \langle \overset{g}{21x^2y^2+4y^2+3y+8}, \overset{h}{14x^3y+8xy+3x+8y} \rangle$, if such an f exists.

$g_y = 42x^2y + 8y + 3$
$h_x = 42x^2y + 8y + 3$
$> g_y = h_x \therefore f$ exists

$\int (21x^2y^2 + 4y^2 + 3y + 8)\,dx = 7x^3y^2 + 4xy^2 + 3xy + 8x$

$\int (14x^3y + 8xy + 3x + 8y)\,dy = 7x^3y^2 + 4xy^2 + 3xy + 4y^2$

$$\boxed{f(x, y) = 7x^3y^2 + 4xy^2 + 3xy + 8x + 4y^2 + C}$$

8. (8 points) Find parametric equations for the tangent line to the surface $z = \overset{9}{x^2} + \overset{+12}{2xy} - \overset{-8}{y^3}$ at the point $(3, 2)$ and in the direction of $\mathbf{v} = \langle 4, 3 \rangle$. $\Rightarrow \vec{u} = \langle \frac{4}{5}, \frac{3}{5} \rangle$

$\nabla z = \langle 2x + 2y, 2x - 3y^2 \rangle \Rightarrow \nabla z(3,2) = \langle 6+4, 6-12 \rangle$
$= \langle 10, -6 \rangle$

$D_{\vec{u}}\, z(2,3) = \langle 10, -6 \rangle \cdot \langle \frac{4}{5}, \frac{3}{5} \rangle = \frac{40}{5} - \frac{18}{5} = \frac{22}{5}$

Tan line

$x = 3 + \frac{4}{5}t \quad y = 2 + \frac{3}{5}t \quad z = 13 + \frac{22}{5}t$

$t \in \mathbb{R}$

9. (14 points) Find the local maxima, local minima, and saddle points of $f(x,y) = x^4 + 4x^2y - 8x^2 + 8y^2 - 16y + 16$.

$f_x = 4x^3 + 8xy - 16x$

$= 4x(x^2 + 2y - 4) = 0$

$x = 0$ or $x^2 = 4 - 2y$

$f_y = 4x^2 + 16y - 16$

$x = 0$ $f_y = 16y - 16 = 0$

$y = 1$

$x^2 = 4 - 2y$ $f_y = 16 - 8y + 16y - 16 = 0$

$8y = 0$

$y = 0$

$\Rightarrow x = \pm 2$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$= (12x^2 + 8y - 16)(16) - 64x^2$

$= 64(3x^2 + 2y - 4 - x^2) = 64(2x^2 + 2y - 4)$

CP	D	$f_{xx} = 4(3x^2 + 2y - 4)$	
(0,1)	$-2(64) < 0$		saddle pt
(2,0)	$4(64) > 0$	$4(8) > 0$ ∪	local min
(-2,0)	$4(64) > 0$	$4(8) > 0$ ∪	local min

10. (10 points) Find the extreme values of $f(x, y) = 2x^2 + y^2 + 2$ subject to the constraint $x^2 + 4y^2 = 4$.

$\nabla f = \langle 4x, 2y \rangle \quad \nabla g = \langle 2x, 8y \rangle$

(1) $4x = 2\lambda x \Rightarrow 4x - 2\lambda x = 2x(2-\lambda) = 0$

$\Rightarrow x = 0$ or $\lambda = 2$ (go to 2)

go to (3)

(2) $2y = 8\lambda y \Rightarrow \lambda = 2 \quad 2y = 16y \Rightarrow y = 0$

(3) $x^2 + 4y^2 = 4 \qquad x = 0 \Rightarrow 4y^2 = 4 \quad y = \pm 1$

$y = 0 \qquad x^2 = 4 \qquad x = \pm 2$

$f(0, \pm 1) = 0 + 1 + 2 = 3 \leftarrow$ min at $(0,1)$ & $(0,-1)$

$f(\pm 2, 0) = 8 + 0 + 2 = 10 \leftarrow$ max at $(2,0)$ & $(-2,0)$

11. (8 points) Find the maximum rate of change of $f(x, y) = 4 + x^2 + 3y^2$ at the point $\left(2, -\frac{1}{2}\right)$ and the direction in which it occurs.

$\nabla f = \langle 2x, 6y \rangle$

$\nabla f(2, -1/2) = \langle 4, -3 \rangle$

max rate of change $\|\nabla f\| = \sqrt{16 + 9} = 5$

direction of max rate is $\langle 4, -3 \rangle$ or $\langle \frac{4}{5}, \frac{-3}{5} \rangle$