Exam 2 **Name**______________________________

Math 231-02 **Calculus III** **October 10, 2018**

Guidelines

- **Calculators are not allowed.**
- Read the questions carefully. You have 65 minutes; use your time wisely.
- You may leave your answers in symbolic form, like $\sqrt{3}$ or $\ln(2)$, unless they simplify further like $\sqrt{9} = 3$ or $\cos(3\pi/4) = -\sqrt{2}/2$.
- Put a box around your final answers when relevant.
- Show all steps in your solutions and make your reasoning clear. Answers with no explanation will not receive full credit, even when correct.
- Use the space provided. If necessary, write "see other side" and continue working on the back of the same page.
- $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$ and $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|| \sin\theta$

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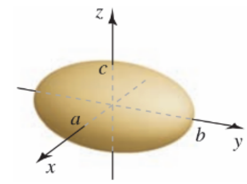
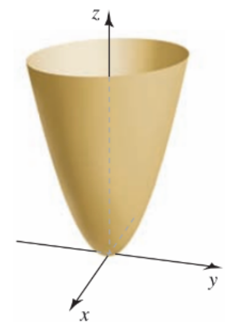
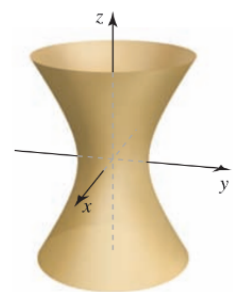
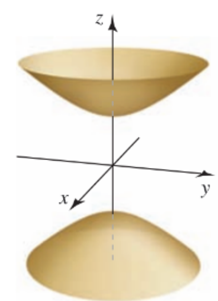
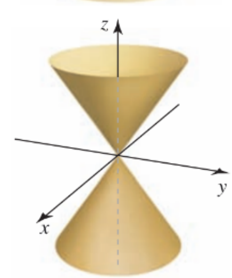
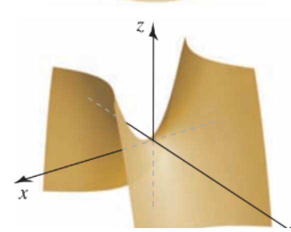
1. (8 points) To be completed once exams are graded and returned. Please correct any problem with points deducted. All corrections should be completed neatly on a separate sheet of paper. Once you have finished your corrections, take your exam and corrections to the Office of Student Learning (OSL), and a tutor will check your answers and sign below. The checked solutions should be given to your instructor.

   Signature: ____________________________
   Print Name: ____________________________
   Date: ______________

Table 12.1

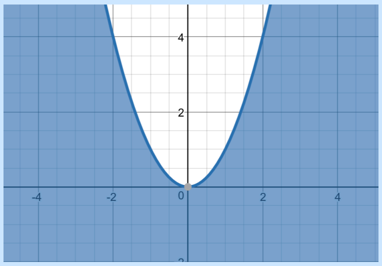
Name	Standard Equation	Features	Graph
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	All traces are ellipses.	
Elliptic paraboloid	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	Traces with $z = z_0 > 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are parabolas.	
Hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Traces with $z = z_0$ are ellipses for all z_0. Traces with $x = x_0$ or $y = y_0$ are hyperbolas.	
Hyperboloid of two sheets	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Traces with $z = z_0$ with $\lvert z_0 \rvert > \lvert c \rvert$ are ellipses. Traces with $x = x_0$ and $y = y_0$ are hyperbolas.	
Elliptic cone	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$	Traces with $z = z_0 \neq 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are hyperbolas or intersecting lines.	
Hyperbolic paraboloid	$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$	Traces with $z = z_0 \neq 0$ are hyperbolas. Traces with $x = x_0$ or $y = y_0$ are parabolas.	

Question	Points	Score
1	8	
2	8	
3	6	
4	6	
5	8	
6	8	
7	8	
8	8	
9	8	
10	14	
11	10	
12	8	
Total:	100	

2. (8 points) Find and sketch the domain of $f(x, y) = \sqrt{x^2 - y}$.

Solution:

$x^2 - y \geq 0 \Rightarrow y \leq x^2$, so $D = \{(x, y) : y \leq x^2\}$



3. (6 points) Find the limit $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x - y}{x^2 - y^2}$.

Solution:

$$\lim_{(x,y)\to(1,1)} \frac{x - y}{x^2 - y^2} = \lim_{(x,y)\to(1,1)} \frac{x - y}{(x - y)(x + y)} = \lim_{(x,y)\to(1,1)} \frac{1}{x + y} = \frac{1}{2}$$

4. (6 points) Show that the limit $\displaystyle\lim_{\substack{(x,y)\to(0,0) \\ y\neq x^2}} \frac{y}{x^2 - y}$ does not exist.

Solution:

Suppose we go through the point (0,0) with a parabola $y = mx^2$ with $m \neq 1$ then

$$\lim_{\substack{(x,y)\to(0,0) \\ y\neq x^2}} \frac{y}{x^2 - y} = \lim_{(x,y)\to(0,0)} \frac{mx^2}{x^2 - mx^2} = \lim_{(x,y)\to(0,0)} \frac{mx^2}{x^2(1 - m)} = \lim_{(x,y)\to(0,0)} \frac{m}{1 - m} = \frac{m}{1 - m}$$

for $m \neq 1$, then the limit changes for different values of m so two paths give different limits and the limit does not exist.

Note: any two paths will work For instance suppose $y = 2x$, the $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x}{x^2 - 2x} = \lim_{(x,y)\to(0,0)} \frac{2}{x - 2} = -1$ and go through $(0, 0)$ throught $y = 0$ the $\displaystyle\lim_{(x,y)\to(0,0)} \frac{0}{x^2 - 0} = 0$ so again two paths give different limits and the limit does not exist.

5. (8 points) If $w = xe^{yz}$, compute $\dfrac{\partial^3 w}{\partial y \partial x \partial z}$.

Solution:

Start by taking the first derivative with respect to x: $w_x = e^{yz}$.

Now take the derivative of w_x with respect to y: $w_{xy} = ze^{yz}$.

Finally, take the derivative of w_{xy} with respect to z: $w_{xyz} = e^{yz} + yze^{yz}$

The ordering xyz is not important since the function is continuous, they can do it any ordering.

Dr. Wolfords class only had to do w_{xy}.

6. (8 points) Find an equation of the tangent plane to surface $x^2 + 2xy - y^2 + z^2 = 7$ at the point $P(1, -1, 3)$.

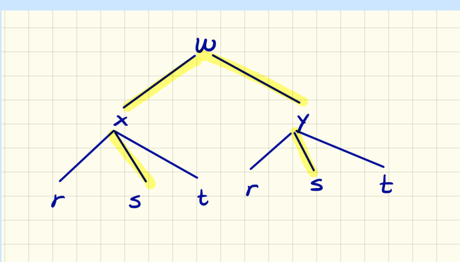
Solution:

$F(x, y, z) = x^2 + 2xy - y^2 + z^2 - 7 = 0$ so $\nabla F = \langle 2x + 2y, 2x - 2y, 2z \rangle$ And the normal vector to the plane $\mathbf{n} = \nabla F(1, -1, 3) = \langle 0, 4, 6 \rangle$.

So the plane is $4(y + 1) + 6(z - 3) = 0 \Leftrightarrow 4y + 6z = 14$

7. (8 points) Write a Chain Rule formula for $\dfrac{\partial w}{\partial s}$ for $w = g(x, y)$, $x = h(r, s, t)$, and $y = k(r, s, t)$.

Solution:



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

8. (8 points) Find the derivative of the function $\cos(xy) + e^{yz} + \ln(xz)$ at the point $P(1, 0, 1/2)$ in the direction $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.

Solution:

$\nabla f = \left\langle -y\sin(xy) + \frac{1}{x}, -x\sin(xy) + ze^{yz}, ye^{yz} + \frac{1}{z} \right\rangle$ and

$\nabla f(1, 0, 1/2) = \left\langle 1, \frac{1}{2}, 2 \right\rangle$

And the unit vector for the directions is $\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{9}} \langle 2, 1, -2 \rangle = \left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right\rangle$

So $D_{\mathbf{u}} f(1, 0, 1/2) = \left\langle 1, \frac{1}{2}, 2 \right\rangle \cdot \left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right\rangle = \frac{2}{3} + \frac{1}{6} - \frac{4}{3} = -\frac{3}{6} = -\frac{1}{2}$

9. (8 points) Is $f(x, y) = x^2 - xy + y^2 - 3$ more sensitive to changes in x or to changes in y when it is near the point $(1, 2)$? Use differentials to justify your answer.

Solution:

$\Delta z \approx dz = f_x(1, 2)dx + f_y(1, 2)dy$.

Now $f_x = 2x - y$ and $f_y = -x + 2y$, so $f_x(1, 2) = 0$ and $f_y(1, 2) = 3$

Thus $\Delta z \approx 0dx + 3dy$ and $f(x, y)$ is more sensitive to changes in the y-direction.

10. (14 points) Find the local maxima, local minima, and saddle points of $f(x, y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$.

Solution:

$f_x = 6x^2 - 18x \qquad f_x = 6x(x-3) = 0 \Rightarrow x = 0$ or $x = 3$

$f_y = 6y^2 + 6y - 12 \qquad f_y = 6(y+2)(y-1) = 0 \Rightarrow y = -2$ or $y = 1$

Critical Points are: $(0, -2),\ (0, 1),\ (3, -2),\ (3, 1)$

To classify: $f_{xx} = 12x - 18 = 6(2x - 3)$, $f_{yy} = 12y + 6 = 6(2y + 1)$, and $f_{xy} = 0$

So $D(x, y) = f_{xx}f_{yy} - (f_{xy})^2 = 36(2x - 3)(2y + 1)$

CP	$D(x, y)$	f_{xx}	Classification
$(0, -2)$	$D > 0$ (+)	$f_{xx} < 0$ (-)	local maximum
$(0, 1)$	$D < 0$ (-)	$f_{xx} < 0$ (-)	saddle point
$(3, -2)$	$D < 0$ (-)	$f_{xx} > 0$ (-)	saddle point
$(3, 1)$	$D > 0$ (+)	$f_{xx} > 0$ (+)	local minimum

11. (10 points) Use Lagrange multipliers to find the maximum and minimum values of $f(x, y, z) = x^2y^2z$, assuming $x \geq 0$ and $y \geq 0$, subject to the constraint $2x^2 + y^2 + z^2 = 25$.

Solution:

$\nabla f = \lambda \nabla g$

Our equations are:

$$\begin{aligned} 2xy^2z &= 4\lambda x \\ 2x^2yz &= 2\lambda y \\ x^2y^2 &= 2\lambda z \end{aligned}$$

So $2x^2y^2z = 4\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$ which gives

$2\lambda(2x^2 - y^2) = 0 \Rightarrow 2x^2 = y^2$ or $x^2 = \dfrac{1}{2}y^2$

$2\lambda(2z^2 - y^2) = 0 \Rightarrow 2z^2 = y^2$ or $z^2 = \dfrac{1}{2}y^2$

Substituting into the constraint:

$$2\left(\frac{1}{2}y^2\right) + y^2 + \frac{1}{2}y^2 = 25 \quad \Rightarrow \quad \frac{5}{2}y^2 = 25 \quad \Rightarrow \quad y^2 = 10 \Rightarrow y = \sqrt{10}$$

So $x^2 = \dfrac{1}{2}10 \Rightarrow x = \sqrt{5}$ and $z^2 = \dfrac{1}{2}10 \Rightarrow z = \pm\sqrt{5}$

Critical points are $(\sqrt{5}, \sqrt{10}, \sqrt{5})$ and $(\sqrt{5}, \sqrt{10}, -\sqrt{5})$

$f(\sqrt{5}, \sqrt{10}, \sqrt{5}) = 50\sqrt{5}$, max f and $f(\sqrt{5}, \sqrt{10}, -\sqrt{5}) = -50\sqrt{5}$, min f.

12. (8 points) Find the direction in which the function $h(x, y, z) = \ln(x^2+y^2-1)+y+6z$ increase and decrease the most rapidly at $P(1, 1, 0)$. What is the rate of change in these directions?

Solution:

$\nabla h = \left\langle \dfrac{2x}{x^2 + y^2 - 1}, \dfrac{2y}{x^2 + y^2 - 1} + 1, 6 \right\rangle$

$\nabla h(1, 1, 0) = \langle 2, 3, 6 \rangle$

Increases most rapidly in $\langle 2, 3, 6 \rangle$ direction or $\dfrac{1}{\sqrt{49}} \langle 2, 3, 6 \rangle = \dfrac{1}{7} \langle 2, 3, 6 \rangle$ with a rate of change of $\sqrt{49} = 7$. Decreases most rapidly in $\langle -2, -3, -6 \rangle$ direction or

$\frac{1}{\sqrt{49}} \langle -2, -3, -6 \rangle = \frac{1}{7} \langle -2, -3, -6 \rangle$ with a rate of change of $-\sqrt{49} = -7$.