Exam 1 | **Name** Solution

Math 231-01 | **Calculus III** | **February 14, 2018**

Guidelines

- **Calculators are not allowed.**
- Read the questions carefully. You have 65 minutes; use your time wisely.
- You may leave your answers in symbolic form, like $\sqrt{3}$ or $\ln(2)$, unless they simplify further like $\sqrt{9} = 3$ or $\cos(3\pi/4) = -\sqrt{2}/2$.
- Put a box around your final answers when relevant.
- Show all steps in your solutions and make your reasoning clear. Answers with no explanation will not receive full credit, even when correct.
- Use the space provided. If necessary, write "see other side" and continue working on the back of the same page.
- $\vec{u} \cdot \vec{v} = ||\vec{u}||||\vec{v}|| \cos\theta$ and $||\vec{u} \times \vec{v}|| = ||\vec{u}||||\vec{v}|| \sin\theta$

Question	Points	Score
1	8	
2	12	
3	12	
4	12	
5	6	
6	6	
7	12	
8	8	
9	12	
10	12	
11	0	
Total:	100	

1. (8 points) Complete test corrections.

2. Let $\vec{PQ}$ extend from $P(2,0,6)$ and $Q(2,-8,5)$

a. (4 points) Find the position vector $\vec{PQ}$.

$\vec{PQ} = \langle 2-2, -8-0, 5-6 \rangle = \langle 0, -8, -1 \rangle$

b. (4 points) Find the the midpoint M on the line segment PQ. Then find the magnitude of $\vec{PM}$.

$M = \left(\frac{2+2}{2}, \frac{-8+0}{2}, \frac{5+6}{2}\right) = (2, -4, 11/2)$

$\vec{PM} = \langle 0, -4, -1/2 \rangle$

$\|\vec{PM}\| = \sqrt{16 + 1/4} = \frac{\sqrt{65}}{2}$

c. (4 points) Find a vector of length 8 with direction opposite of that of $\vec{PQ}$.

$\frac{-8}{\sqrt{65}} \langle 0, -8, -1 \rangle$

3. Let $\vec{u} = \langle 0, 3, -4 \rangle$ and $\vec{v} = \langle -5, 6, 0 \rangle$

$\|\vec{u}\| = \sqrt{9+16} = 5 \quad \|\vec{v}\| = \sqrt{25+36} = \sqrt{61}$

a. (4 points) Compute $\vec{u} \cdot \vec{v}$.

$\vec{u} \cdot \vec{v} = 0 + 18 + 0 = 18$

b. (4 points) Find the angle between $\vec{u}$ and $\vec{v}$.

$\cos\theta = \frac{18}{5\sqrt{61}} \Rightarrow \theta = \cos^{-1}\left(\frac{18}{5\sqrt{61}}\right)$

c. (4 points) Find $\text{proj}_u v$

$= \frac{18}{25} \langle 0, 3, -4 \rangle$

4. Consider three points $P(2,-3,5)$, $Q(3,1,0)$ and $R(-1,0,7)$

a. (4 points) Find an equation for the line containing P and Q.

$\vec{v} = \vec{PQ} = \langle 1, 4, -5 \rangle$

$\vec{r}(t) = \langle 2, -3, 5 \rangle + t\langle 1, 4, -5 \rangle = \langle 2+t, -3+4t, 5+5t \rangle$

b. (4 points) Find an equation for the plane containing P, Q, and R.

$\vec{PQ} = \langle 1, 4, -5 \rangle$

$\vec{PR} = \langle -3, 3, 2 \rangle$

$\vec{N} = \vec{PQ} \times \vec{PR} = \langle 8+15, -(2-15), 3+12 \rangle = \langle 23, 13, 15 \rangle$

$23(x-2) + 13(y+3) + 15(z-5) = 0$

OR

$23x + 13y + 15z = 82$

c. (4 points) Find the area of the triangle determined by P, Q, and R.

$\frac{1}{2}\|\vec{N}\| = \frac{1}{2}\sqrt{23^2 + 13^2 + 15^2} = \frac{1}{2}\sqrt{923}$

5. (6 points) Find the equation of the sphere with center $(2, -3, 4)$ and tangent to the xz-plane.

$(2,0,4)$

$r = 3 = \sqrt{(2-2)^2+(-3-0)^2+(4-4)^2}$

$(x-2)^2 + (y+3)^2 + (z-4) = 9$

6. (6 points) Find the $\lim_{t\to -3} \left\langle \frac{|t+3|}{t+3}, \frac{|t+3|}{t-3} \right\rangle$, if it exists. If it does not exist, explain why.

$\lim_{t\to -3^-} \frac{-(t+3)}{t+3} = -1 \neq \lim_{t\to -3^+} \frac{t+3}{t+3} = 1 \quad \therefore \lim_{t\to -3} \frac{|t+3|}{t+3}$ DNE

which implies $\lim_{t\to -3} \vec{r}(t)$ DNE

7. (12 points) For the vector curve $\vec{r}(t) = \langle \sin t, 2\sin t, \sqrt{5}\cos t\rangle$, find the unit tangent vector, the principle unit normal vector and the equation of the osculating plane(the plane containing the unit tangent vector and the principle unit normal vector).

$\vec{r}\,' = \langle \cos t, 2\cos t, -\sqrt{5}\sin t \rangle$

$\|\vec{r}\,'\| = \sqrt{\cos^2 t + 4\cos^2 t + 5\sin^2 t} = \sqrt{5}$

$$\boxed{\vec{T}(t) = \left\langle \frac{1}{\sqrt{5}}\cos t, \frac{2}{\sqrt{5}}\cos t, -\sin t \right\rangle}$$

$\vec{T}\,'(t) = \left\langle -\frac{1}{\sqrt{5}}\sin t, \frac{-2}{\sqrt{5}}\sin t, -\cos t \right\rangle$

$\|\vec{T}\,'\| = \sqrt{\frac{1}{5}\sin^2 t + \frac{4}{5}\sin^2 t + \cos^2 t} = 1$

$$\boxed{\vec{N} = \left\langle -\frac{1}{\sqrt{5}}\sin t, \frac{-2}{\sqrt{5}}\sin t, -\cos t \right\rangle}$$

$\vec{B} = \vec{T} \times \vec{N} = \left\langle \frac{-2}{\sqrt{5}}\cos^2 t - \frac{2}{\sqrt{5}}\sin^2 t, -\left(-\frac{1}{\sqrt{5}}\cos^2 t - \frac{1}{\sqrt{5}}\sin^2 t\right), 0 \right\rangle$

$= \left\langle -\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0 \right\rangle$

O. Plane at $t = t_0$

$-\frac{2}{\sqrt{5}}(x - \sin(t_0)) + \frac{1}{\sqrt{5}}(y - 2\sin t_0) + 0(z - \sqrt{5}\cos t_0) = 0$

$$\Rightarrow \boxed{\frac{-2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y = 0}$$

8. (8 points) Find the arc length of the curve $\vec{r}(t) = \left\langle t^2, \frac{4\sqrt{2}}{3}t^{3/2}, 2t \right\rangle$ for $1 \le t \le 3$.

$\vec{r}\,' = \langle 2t, 2\sqrt{2}\, t^{1/2}, 2 \rangle$ $\quad \|\vec{r}\,'\| = \sqrt{4t^2+8t+4} = \sqrt{4(t^2+2t+1)}$

$= 2(t+1)$

$L = \int_1^3 (2t+2)\, dt = \left[t^2+2t\right]_1^3 = 9-1+2(3-1) = 8+4 = \boxed{12}$

9. (12 points) Given $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t) = \langle \sin 3t, t^2, \cos 3t \rangle$ and $\vec{v}(0) = \langle 0, 0, 0 \rangle$, and $\vec{r}0 = \langle 2, -5, 3 \rangle$, find the position vector $\vec{r}(t)$. $v(t) = \int a(t)\,dt = \langle -\frac{1}{3}\cos 3t + C_1, \frac{t^3}{3} + C_2, \frac{1}{3}\sin 3t + C_3 \rangle$

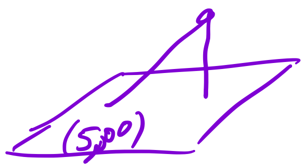
$v(0) = \langle -\frac{1}{3} + C_1, C_2, C_3 \rangle = \langle 0, 0, 0 \rangle \Rightarrow v(t) = \langle -\frac{1}{3}\cos 3t + \frac{1}{3}, \frac{t^3}{3}, \frac{1}{3}\sin 3t \rangle$

$\vec{r} = \int v(t)\,dt = \langle -\frac{1}{9}\sin 3t + \frac{1}{3}t + D_1, \frac{t^4}{12} + D_2, -\frac{1}{3}\cos 3t + D_3 \rangle$

$\vec{r}(t) = \langle -\frac{1}{9}\sin 3t + \frac{1}{3}t + 2, \frac{t^4}{12} - 5, -\frac{1}{9}\cos 3t + \frac{28}{9} \rangle$

10. Let P be a point with coordinates $(1, 2, -3)$, $\mathcal{L}$ be the line with equation $\vec{r}(t) = \langle 2 + 5t, -3 + t, -4t \rangle$ and $\mathcal{P}$ be the plane with equation x-3y+4z=5. Find the following:

a. (4 points) the distance from P to $\mathcal{P}$.



$\vec{u} = PP_0 = \langle 4, -2, 3 \rangle$

$\vec{N} = \langle 1, -3, 4 \rangle$

$d = \frac{\vec{u} \cdot \vec{N}}{\|\vec{N}\|} = \frac{22}{\sqrt{26}}$

b. (4 points) the point at which $\mathcal{L}$ intersects $\mathcal{P}$.

$2+5t-3(-3+t)+4(-4t) = 5 \Rightarrow t = \frac{3}{7}$

pt $\left(\frac{29}{7}, -\frac{18}{7}, -\frac{12}{7}\right)$

c. (4 points) than angle at which $\mathcal{L}$ intersects $\mathcal{P}$

$\vec{v} = \langle 5, 1, -4 \rangle$

$\vec{N} = \langle 1, -3, 4 \rangle$

$\theta = \cos^{-1}\left(\frac{5-3-16}{\sqrt{42}\sqrt{26}}\right) - \frac{\pi}{2}$

$= \cos^{-1}\left(\frac{-14}{\sqrt{42}\sqrt{26}}\right) - \pi/2$

11. (4 points (bonus)) Let $\vec{v} = \langle a, b, c \rangle$ and $\vec{w} = \langle \alpha, \beta, \gamma \rangle$. Give conditions on the constants a, b, c, α, β, and γ that guarantees that

a. $\vec{v}$ is parallel to $\vec{w}$

$\alpha = ka$, $\beta = kb$, $\gamma = kc$ $\quad K$ is a real #.

b. $\vec{v}$ is perpendicular to $\vec{w}$

$a\alpha + b\beta + c\gamma = 0$