Exam 1 **Name**____

Math 231-03 **Calculus III** **September 14, 2018**

Guidelines

- **Calculators are not allowed.**
- Read the questions carefully. You have 65 minutes; use your time wisely.
- You may leave your answers in symbolic form, like $\sqrt{3}$ or $\ln(2)$, unless they simplify further like $\sqrt{9} = 3$ or $\cos(3\pi/4) = -\sqrt{2}/2$.
- Put a box around your final answers when relevant.
- Show all steps in your solutions and make your reasoning clear. Answers with no explanation will not receive full credit, even when correct.
- Use the space provided. If necessary, write ”see other side” and continue working on the back of the same page.
- $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos\theta$ and $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||||\mathbf{v}|| \sin\theta$
- $\kappa(t) = \dfrac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \dfrac{1}{|\mathbf{v}|}\left|\dfrac{d\mathbf{T}}{dt}\right|$

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1. (8 points) To be complete once exams are graded and returned. Please correct any problem with points deducted. All corrections should be completed neatly on a separate sheet of paper. Once you have finished your corrections, take your exam and corrections to the Math Department Help Room, Weir 220, and a tutor will check your answers and sign below. The checked solutions should be given to your instructor.

   Signature: ________________________________
   Print Name: ________________________________
   Date: ________________

Question	Points	Score
1	8	
2	20	
3	4	
4	12	
5	8	
6	8	
7	8	
8	16	
9	8	
10	8	
Total:	100	

2. Let $\mathbf{u} = \langle 3, -5, 2\rangle$ and $\mathbf{v} = \langle -9, 5, 1\rangle$, find

   a. (4 points) Find $\mathbf{v} \cdot \mathbf{u}$.

   Solution:
   $\mathbf{v} \cdot \mathbf{u} = -9(3) + 5(-5) + 1(2) = -27 - 25 + 2 = -50$

   b. (4 points) Find the angle between the vectors, $\mathbf{u}$ and $\mathbf{v}$.

   Solution:
   $\theta = \cos^{-1}\left(\dfrac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}||\mathbf{u}|}\right)$
   From part a, $\mathbf{v} \cdot \mathbf{u} = -50$, $|\mathbf{v}| = \sqrt{81 + 25 + 1} = \sqrt{107}$, and $|\mathbf{u}| = \sqrt{9 + 25 + 4} = \sqrt{38}$ so

   $$\theta = \cos^{-1}\left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{v}||\mathbf{u}|}\right) = \cos^{-1}\left(\frac{-50}{\sqrt{38}\sqrt{107}}\right)$$

   c. (6 points) Find two unit vectors that are orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

   Solution:
   The vector $\mathbf{u} \times \mathbf{v}$ will be orthogonal to both $\mathbf{v}$ and $\mathbf{u}$,

   $$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -5 & 2 \\ -9 & 5 & 1 \end{vmatrix} = \langle -5 - 10, -(3 + 18), 15 - 45\rangle = \langle -15, -21, -30\rangle = -3\langle 5, 7, 10\rangle$$

   Two unit vectors orthogonal to $\mathbf{u}$ and $\mathbf{v}$ is $\pm\dfrac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|} = \pm\dfrac{1}{\sqrt{174}}\langle 5, 7, 10\rangle$, since $|\mathbf{u} \times \mathbf{v}| = 3\sqrt{25 + 49 + 100} = 3\sqrt{174}$

   d. (6 points) The projection of $\mathbf{u}$ onto $\mathbf{v}$.

   Solution:
   $$proj_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}\vec{v} = \frac{-50}{81 + 25 + 1}\langle -9, 5, 1\rangle = \frac{-50}{107}\langle -9, 5, 1\rangle$$

3. (4 points) Find the values of x such that the vectors $\langle 3, 2, x\rangle$ and $\langle 2x, 4, x\rangle$ are orthogonal.

   Solution:
   Two vectors are orthogonal is $\mathbf{u} \cdot \mathbf{v} = 0$, for our problem $\mathbf{u} \cdot \mathbf{v} = 3(2x) + 2(4) + x(x) = x^2 + 6x + 8 = (x + 2)(x + 4) = 0$ if $x = -2$ or $x = -4$.

4. (12 points) Find a line that is perpendicular to the lines $\mathbf{r} = \langle 4t, 1+2t, 3t\rangle$ and $\mathbf{R} = \langle -1+s, -7+2s, -12+3s\rangle$ and passes through the point of intersection of the lines $\mathbf{r}$ and $\mathbf{R}$.

Solution:

The direction vector for the perpendicular line will be the cross product of the direction vector for each line $\mathbf{v}_r = \langle 4,2,3\rangle$ and $\mathbf{v}_R = \langle 1,2,3\rangle$ is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix} = \mathbf{i}(6-6) - \mathbf{j}(12-3) + \mathbf{k}(8-2) = -9\mathbf{j} + 6\mathbf{k}$

To find the point of intersection of the two lines, solve (1) $4t = -1+s$, (2) $1+2t = -7+s$ and (3) $3t = -12+3s$ from (1) $s = 4t+1$, plug s into (2) $1+2t = -7+2(4t+1)$ which results in $6 = 6t$ thus $t = 1$ and $s = 4+1 = 5$ check that this solution works in (3) $3(1) = -12+15$ which is true. So the point of intersection is $(4,3,3)$.

So the line is $\mathbf{r}(t) = \langle 4,3,3\rangle + t\langle 0,-9,6\rangle = \langle 4, 3-9t, 3+6t\rangle$.

5. (8 points) Find the area of the parallelogram that has two adjacent sides $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution:

The area of the parallelogram is $|\mathbf{u}\times\mathbf{v}|$

$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -2 \\ 3 & 2 & -1 \end{vmatrix} = \mathbf{i}(1+4) - \mathbf{j}(-2+6) + \mathbf{k}(4+3) = 5\mathbf{i} - 4\mathbf{j} + 7\mathbf{k}$

So $A = |\mathbf{u}\times\mathbf{v}| = \sqrt{25+16+49} = \sqrt{90}$.

6. (8 points) Find a parametric equation of the line tangent to the curve $\mathbf{r}(t) = \left\langle \sqrt{2t+1}, \sin \pi t, 4\right\rangle$ at $t = 4$.

Solution:

$\mathbf{r}'(t) = \left\langle \frac{1}{2}(2t+1)^{-1/2}\cdot 2, \pi\cos(\pi t), 0\right\rangle$ so the direction of the tangent line is $\mathbf{r}'(4) = \left\langle \frac{1}{3}, \pi, 0\right\rangle = \mathbf{v}$

The point on the tangent line where it intersects the curve is $\mathbf{R}_0 = \mathbf{r}(4) = \langle 3,0,4\rangle$

line tangent is $r\vec{R} = \mathbf{R}_0 + t\vec{v}$

Parametric equations: $x = 3 + \frac{1}{3}t$, $y = \pi t$, and $z = 4$ for $t \epsilon \mathbb{R}$

7. (8 points) If the velocity of a particle is $\mathbf{v}(t) = \left\langle \frac{t}{t^2+1}, te^{-t^2}, -\frac{2t}{\sqrt{t^2+4}}\right\rangle$, find the position vector $\vec{r}(t)$ if $\mathbf{r}(0) = \left\langle 1, \frac{3}{2}, -3\right\rangle$.

Solution:
$\mathbf{r}(t) = \int \mathbf{v}(t)dt = \left\langle \int \frac{t}{t^2+1}\,dt, \int te^{-t^2}\,dt, \int \frac{-2t}{\sqrt{t^2+4}}\,dt \right\rangle$ (Let $u = t^2+1$ in the first integral, $w = -t^2$ on the second, and $v = t^2+4$ in the third integral)

So $\mathbf{r}(t) = \left\langle \frac{1}{2}\ln|t^2+1| + C_1, -\frac{1}{2}e^{-t^2} + C_2, -2\sqrt{t^2+4} + C_3 \right\rangle$

And since
$\mathbf{r}(0) = \left\langle 1, \frac{3}{2}, -3 \right\rangle = \left\langle \frac{1}{2}\ln 1 + C_1, -\frac{1}{2}e^0 + C_2, -2\sqrt{4} + C_3 \right\rangle = \left\langle C_1, -\frac{1}{2} + C_2, -4 + C_3 \right\rangle$ then

$\mathbf{r}(t) = \left\langle \frac{1}{2}\ln|t^2+1| + 1, -\frac{1}{2}e^{-t^2} + 2, -2\sqrt{t^2+4} + 1 \right\rangle$

8. (16 points) For the vector curve $\mathbf{r}(t) = \langle \cos t, \sin t, \ln \cos t \rangle$, find the unit tangent vector, the principle unit normal vector, the binormal vector and curvature **all at** $t = 0$.

Solution:

$\vec{r}\,'(t) = \left\langle -\sin t, \cos t, \dfrac{1}{\cos t} \cdot -\sin t \right\rangle = \langle -\sin t, \cos t, -\tan t \rangle$ and

$|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t} = \sqrt{1 + \tan^2 t} = \sqrt{\sec^2 t} = \sec t$,

Thus

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{1}{\sec t} \langle -\sin t, \cos t, -\tan t \rangle$$
$$= \langle -\sin t \cos t, \cos^2 t, -\sin t \rangle = \left\langle -\frac{1}{2} \sin 2t, \cos^2 t, -\sin t \right\rangle$$

So $\vec{T}\,'(t) = \langle -\cos 2t, -2\cos t \sin t, -\cos t \rangle$ and $\vec{T}\,'(0) = \langle -1, 0, -1 \rangle$, $|\vec{T}\,'(0)| = \sqrt{1+0+1} = \sqrt{2}$

Finally, $\vec{T}(0) = \langle 0, 1, 0 \rangle$,

$\vec{N}(0) = \dfrac{\vec{T}\,'(0)}{|\vec{T}\,'(0)|} = \dfrac{1}{\sqrt{2}} \langle -1, 0, -1 \rangle = \left\langle -\dfrac{1}{\sqrt{2}}, 0, -\dfrac{1}{\sqrt{2}} \right\rangle$

$\vec{B}(0) = \vec{T}(0) \times \vec{N}(0) = \left\langle -\dfrac{1}{\sqrt{2}}, 0, \dfrac{1}{\sqrt{2}} \right\rangle$

And curvature $\kappa(0) = \dfrac{1}{|\vec{r}\,'(0)|} \left| \vec{T}\,'(0) \right| = \dfrac{1}{\sec 0} \sqrt{2} = \sqrt{2}$.

9. (8 points) Find the arc length of the curve $\mathbf{r}(t) = \left\langle \cos(2t), 2t^{3/2}, \sin(2t) \right\rangle$ for $0 \le t \le 5$.

Solution:

To find arc length $L = \displaystyle\int_a^b |\vec{r}\,'(t)|\,dt$

$\vec{r}\,'(t) = \left\langle -2\sin 2t, 3t^{1/2}, 2\cos 2t \right\rangle$

And $|\vec{r}\,'(t)| = \sqrt{4\sin^2 2t + 9t + 4\cos^2 2t} = \sqrt{4+9t}$

Now arc length

$$L = \int_0^5 \sqrt{4+9t}\,dt = \int_4^{49} \frac{1}{9}\sqrt{u}\,du = \frac{1}{9}\cdot\frac{2}{3}u^{3/2}\Big|_4^{49} = \frac{2}{27}\left(49^{3/2} - 4^{3/2}\right) = \frac{2}{27}(7^3 - 2^3)$$

10. (8 points) Determine if the curve $\mathbf{r}(t) = \langle 13\cos t, 5\sin t, 12\sin t\rangle$ for $0 \le t \le \pi$ uses arc length as a parameter. If not, find a description that uses arc length as a parameter.

Solution:

$\mathbf{r}'(t) = \langle -13\sin t, 5\cos t, 12\cos t\rangle$

So $|\mathbf{r}'(t)| = \sqrt{169\sin^2 t + 25\cos^2 t + 144\cos^2 t} = \sqrt{169(\sin^2 t + \cos^2 t)} = \sqrt{169} = 13 \neq 1$, therefore, it is not parameterized by arc length.

Now $s(t) = \displaystyle\int_0^t 13\,du = 13t$ and $t = \dfrac{s}{13}$.

The description that uses arc length as parameter is

$\mathbf{r}(s) = \left\langle 13\cos\left(\dfrac{s}{13}\right), 5\sin\left(\dfrac{s}{13}\right), 12\sin\left(\dfrac{s}{13}\right)\right\rangle$ for $0 \le s \le 13\pi$.