Final Exam Review **Math 132** **L. Ballou**

1. Evaluate the following:

a. $\int \frac{\sin^3 x}{\sqrt{\cos x}}dx$. $= \int \frac{\sin x\,(1-\cos^2 x)}{\sqrt{\cos x}}dx$ $u = \cos x$, $du = -\sin x\,dx$

$= -\int \frac{(1-u^2)}{\sqrt{u}}du = \int (u^{3/2} - u^{-1/2})\,du = \frac{2}{5}u^{5/2} - 2u^{1/2} + C$

$= \frac{2}{5}\cos^{5/2} x - 2\cos x + C$

b. $\int x^2 \ln x\,dx$. $u = \ln x$ $du = \frac{1}{x}dx$

$V = x^3/3$ $dV = x^2 dx$

$\int x^2 \ln x\,dx = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2 dx = \frac{x^3}{3}\ln 3 - \frac{1}{9}x^3 + C$

c. $\int \frac{2x(x^2-1)}{x^2-4}dx$

$$\begin{array}{r|l} & \quad 2x \\ \hline x^2-4 & 2x^3 - 2x \\ & \underline{2x^3 - 8x} \\ & \quad 6x \end{array}$$

$\frac{6x}{(x-2)(x+2)} = \frac{A}{(x-2)} + \frac{B}{x+2} = \frac{3}{x-2} + \frac{3}{x+2}$

$\int \frac{2x(x^2-1)}{x^2-4}dx = \int \left(2x + \frac{3}{x-2} + \frac{3}{x+2}\right)dx = x^2 + 3\ln|x-2| + 3\ln|x+2| + C$

d. $\int x\cos x\,dx$ $u = x$ $du = dx$

$V = \sin x$ $dV = \cos x\,dx$

$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$

e. $\int \frac{1}{(x-2)(x^2+4)}dx$

$$\frac{1}{(x-2)(x^2+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4}$$

$$1 = A(x^2+4) + (Bx+C)(x-2)$$

$Ax^2 + Bx^2 = 0x^2 \Rightarrow A+B=0 \quad A=-B$

$-2Bx + Cx + 0x \Rightarrow -2B+C=0 \quad C=2B$

$-2C+4A=1 \quad -4B-4B=1 \quad B=-1/8$

$A=1/8 \quad C=-1/4$

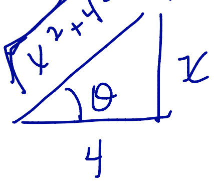
$$\int \frac{1}{(x-2)(x^2+4)}dx = \int\left[\frac{1}{8}\,\frac{1}{x-2} - \frac{1}{8}\,\frac{x}{x^2+4} - \frac{1}{4}\cdot\frac{1}{x^2+4}\right]dx$$

$$= \frac{1}{8}\ln|x-2| - \frac{1}{16}\ln|x^2+4| - \frac{1}{8}\arctan\left(\frac{x}{2}\right) + C$$

$x = 4\tan\theta \quad dx = 4\sec^2\theta\, d\theta$

f. $\int \frac{dx}{\sqrt{x^2+16}}$

$$= \int \frac{4\sec^2\theta\, d\theta}{\sqrt{16\tan^2\theta+16}} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$$

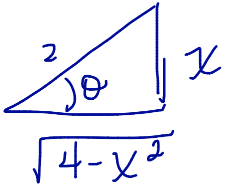


$$= \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C$$

$x = 2\sin\theta \quad dx = 2\cos\theta\, d\theta$

g. $\int \frac{x^2}{\sqrt{4-x^2}}dx$.

$$= \int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\cdot 2\cos\theta\, d\theta = \int 4\sin^2\theta\, d\theta = 2\int 1-\cos 2\theta\, d\theta$$



$$= 2\left[\theta - \frac{1}{2}\sin 2\theta\right] = 2\theta - 2\sin\theta\cos\theta$$

$$= 2\arcsin\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + C$$

h. $\int \tan^3 x \sec^3 x dx$

$$= \int \tan^2 x \sec^2 x \tan x \sec x\, dx = \int (\sec^2 x - 1)\sec^2 x \sec x \tan x\, dx$$

$u = \sec x$
$du = \sec x \tan x\, dx$

$$= \int (u^2-1)u^2\, du = \int (u^4 - u^2)\, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$$

i. $\int_2^\infty \frac{dx}{x\ln x} = \lim_{B\to\infty} \int_2^B \frac{dx}{x \ln x} = \lim \ln(\ln x)\Big|_2^B$

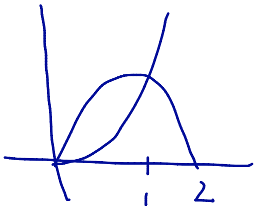
$= \lim_{B\to\infty} (\ln \ln B - \ln \ln 2) = \infty$

$\therefore$ integral divergs.

2. Let R be the region in the first quadrant bounded by the curve $y = x^3$ and $y = 2x - x^2$. Determine the volume of the solid obtained by revolving R about

   a. The x-axis.

$x^3 = 2x - x^2 \rightarrow x^3 + x^2 - 2x = x(x+2)(x-1)$



$V = \pi \int_0^1 \left[(2x - x^2)^2 - (x^3)^2\right] dx$

   b. The y-axis.

$2\pi \int_0^1 x\left[2x - x^2 - x^3\right] dx$

   c. The line $x = 2$

$2\pi \int_0^1 (2-x)(2x - x^2 - x^3)\, dx$

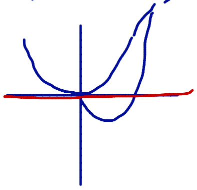
   d. The line $y = -2$.

$\pi \int_0^1 \left[(2x - x^2 - (-2))^2 - (x^3 - (-2))^2\right] dx$

$2x(x-2)$

3. Find the area of the region in the first quadrant bounded by $y = x^2$, $y = 2x^2 - 4x$ and $y = 0$.

$x^2 = 2x^2 - 4x \Rightarrow 0 = x^2 - 4x > 0 = x(x-4)$



$$A = \int_0^2 (x^2 - 0)\,dx + \int_2^4 [x^2 - (2x^2 - 4x)]\,dx$$

4. Find the length of the curve $y = x^{1/2} - \frac{x^{3/2}}{3}$ for $1 \le x \le 3$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{1}{3}\cdot\frac{3}{2}x^{1/2} = \frac{1}{2\sqrt{x}} - \frac{\sqrt{x}}{2}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4x} - \frac{1}{2} + \frac{x}{4} \longrightarrow \left(\frac{dy}{dx}\right)^2 + 1 = \frac{1}{4x} + \frac{1}{2} + \frac{x}{4}$$

$$L = \int_1^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_1^3 \sqrt{\left(\frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2}\right)^2}\,dx$$

5. Find the surface area when the curve $y = x^2$ for $1 \le x \le 2$ is rotated about the y-axis.

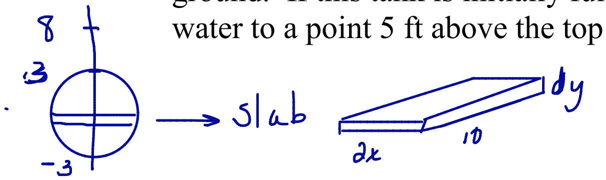
$$\frac{dy}{dx} = 2x \qquad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + 4x^2$$

$$S = \int_1^2 2\pi r\sqrt{1 + (dy/dx)^2}\,dx = \int_1^2 2\pi x\sqrt{1 + 4x^2}\,dx.$$

Note About x-axis:

$$\int_1^2 2\pi x^2\sqrt{1 + 4x^2}\,dx$$

6. A cylindrical tank of radius 3 ft and length of 10 ft is lying on its side on horizontal ground. If this tank is initially full of water, how much work is done to pump all the water to a point 5 ft above the top of the tank?



$x = \sqrt{9-y^2}$

$\partial W = \Delta F \cdot dist$

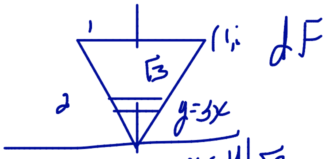
$\rho \cdot Vol \cdot (8-y)$

$\rho \cdot 2x \cdot 10 \cdot \partial y \ (8-y)$

$\rho \cdot 20\sqrt{9-y^2}\ (8-y)\, dy$

$$W = 20\rho \int_{-3}^{3} \sqrt{9-y^2}\ (8-y^2)\, dy$$

7. A trough has vertical ends that are equilateral triangles (downward pointing) with sides of length 2 ft. If the trough is filled will water, find the force exerted by the water on one end of the trough.



$x = y/\sqrt{3}$

$dF = w\cancel{A}d$

$= w(2x)dy(\sqrt{3}-y) = w \cdot 2(y/\sqrt{3})\, dy\, (\sqrt{3}-y)$

$$F = \int_0^{\sqrt{3}} \frac{2}{\sqrt{3}} w \cdot y(\sqrt{3}-y)\, dy.$$

8. Find the Taylor series for $f(x) = \dfrac{1}{4x-3}$ at $a = 1$.

k		
0	$(4x-3)^{-1}$	1
1	$-1(4x-3)^{-2}4$	$-1 \cdot 4$
2	$-1(-2)(4x-3)^{-3}4^2$	$1 \cdot 2 \cdot 4^2$
3	$-1(-2)(-3)(4x-3)^{-4}4^3$	$-1 \cdot 2 \cdot 3\ 4^3$
4	$-1(-2)(-3)(-4)(4x-3)^{-5}4^4$	$4!\ 4^4$

$f^{(K)}(1) = (-1)^K K!\, 4^K$

$$f(x) = \sum_{K=0}^{\infty} \frac{(-1)^K K!\, 4^K}{K!} (x-1)^K$$

$$= \sum_{K=0}^{\infty} (-1)^K 4^K (x-1)^K$$

9. Find the radius and interval of convergence for the power series $\sum_{k=1}^{\infty} \frac{(2x+3)^k}{k^2 4^k}$.

$\lim_{k\to\infty} \left| \frac{(2x+3)^{k+1}}{(k+1)^2 4^{k+1}} \cdot \frac{k^2 4^k}{(2x+3)^k} \right| = \frac{|2x+3|}{4} \lim \frac{k^2}{(k+1)^2} = \frac{|2x+3|}{4} < 1$ conv.

$\Rightarrow -4 < 2x+3 < 4 \Rightarrow -7 < 2x < 1 \Rightarrow -7/2 < x < 1/2$

$x = \frac{1}{2}$ $\sum \frac{1}{k^2}$ p-series conv, $p=2$

$x = -7/2$ $\sum \frac{(-1)^k}{k^2}$, conv b/c $\sum |a_k|$ con

So IOC $[-7/2, 1/2]$

Rad. 2

10. Determine whether the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k^2}{k^3+1}$ converges absolutely, converges conditionally or diverges.

$\sum |a_k| = \sum \frac{k^2}{k^3+1}$ looks like $\sum \frac{k^2}{k^3} = \sum \frac{1}{k}$ div p-series

& since $\lim \frac{k^2/k^3+1}{1/k} = 1$ then by LCT $\sum k^2/k^3+1$ div as well.

However, $\lim \frac{k^2}{k^3+1} = 0$ & $a' = \frac{2k(k^3+1) - k^2(3k^2)}{(k^3+1)^2} = \frac{-k^4+k}{(k^3+1)^2} < 0$ $\forall k > 1$

so a is dec so by AST the series is conv. thus $\sum \frac{(-1)^k k^2}{k^3+1}$ conv cond.

11. Determine whether the series $\sum_{k=1}^{\infty} \frac{2k+3}{k^2+3k+1}$ converges or diverges.

looks like $\sum \frac{2k}{k^2} = 2\sum \frac{1}{k}$ divergent p-series

& $\lim \frac{2k+3}{k^2+3k+1} \cdot \frac{k}{2} = 1$ So by LCT the series diverges.

12. Determine whether the series $\sum_{k=1}^{\infty} \frac{\sin(5k)}{1+3^k}$ converges or diverges.

We know $0 \le |\sin 5k| \le 1$ $\forall k \Rightarrow 0 \le \frac{|\sin 5k|}{1+3^k} \le \frac{1}{1+3^k} \le \frac{1}{3^k}$

& $\sum \frac{1}{3^k}$ is conv geometric series $r = \frac{1}{3}$

so by the comp. test $\sum \frac{|\sin 5k|}{1+3^k}$ conv which implies $\sum \frac{\sin 5k}{1+3^k}$ conv.

13. Find and sketch the four roots of $3+3i$.

$3+3i = 18^{1/2}\,\text{cis}(\pi/4 + 2K\pi)$

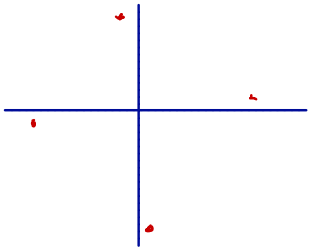
$r = (18^{1/2})^{1/4} = 18^{1/8}$ $\quad \phi = \frac{1}{4}\left(\frac{\pi}{4} + 2K\pi\right) = \frac{\pi}{16} + \frac{K\pi}{2}$

$w_1 = 18^{1/8}\,\text{cis}(\pi/16)$

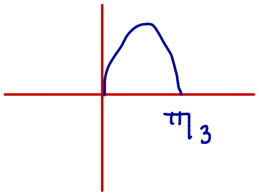
$w_2 = 18^{1/8}\,\text{cis}(9\pi/16)$

$w_3 = 18^{1/8}\,\text{cis}(17\pi/16)$

$w_4 = 18^{1/8}\,\text{cis}(25\pi/16)$

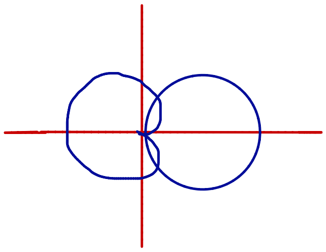


14. Find the area of the region enclosed by one loop of the curve $r = 3\sin 3\theta$.



$$A = \int_0^{\pi/3} \frac{1}{2}\left(3\sin(3\theta)\right)^2 d\theta$$

15. Find the area of the region within both of the curves $r = \cos\theta$ and $r = 1 - \cos\theta$.



$\cos\theta = 1 - \cos\theta \rightarrow 2\cos\theta = 1 \quad \cos\theta = 1/2$

$\theta = \pi/3$

$$2\cdot\frac{1}{2}\int_0^{\pi/3}(1-\cos\theta)^2 d\theta + 2\frac{1}{2}\int_{\pi/3}^{\pi/2}(\cos\theta)^2 d\theta$$

16. Evaluate the expression

a. $13e^{i\pi/2}$ $= 0 + 13i$

b. $\frac{5+3i}{4-2i} \cdot \frac{4+2i}{4+2i} = \frac{14+22i}{20} = \frac{14}{20} + \frac{22}{20}i = \frac{7}{10} + \frac{11}{10}i$

c. $(-1-i)^{24}$ $= \left[\sqrt{2}\ \text{cis}(5\pi/4)\right]^{24} = \left[2^{1/2}\right]^{24} \text{cis}\left(\frac{5\pi}{4} \cdot 24\right)$

$= 2^{12} \cdot \text{cis}(30\pi) = 2^{12}$

d. $|-3-4i|$ $= \sqrt{(-3-4i)(-3+4i)} = \sqrt{9+16} = 5$

17. For the parametric curve $x = t^2$, $y = 3\ln t + 2$, write the equation of the line tangent to the curve at $t = 1$.

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3/t}{2t}\Big|_{t=1} = \frac{3}{2}$ $y - 2 = \frac{3}{2}(x-1)$

18. Find the length of the parametric curve $x = \overset{2t^{3/2}}{\cancel{t^2}} + 1$, $y = 3t$ for $0 \le t \le 2$.

$L = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(dy/dt\right)^2}\,dt = \int_0^2 \sqrt{9t + 9}\ dt$

$3 \cdot \frac{2}{3}(t+1)^{3/2}\Big|_0^2 = 2\left(3^{3/2} - 1\right)$