1. Determine whether or not the sequence $\{a_k\}$ converges and find its limit if it does converge.

a. $a_k = \dfrac{8k-7}{7k-8}$

Solution: $\lim\limits_{k\to\infty} \dfrac{8k-7}{7k-8} = \dfrac{8}{7}$, therefore the sequence converges.

b. $a_k = \dfrac{k-e^k}{k+e^k}$

Solution: $\lim\limits_{k\to\infty} \dfrac{k-e^k}{k+e^k} \overset{L}{=} \lim\limits_{k\to\infty} \dfrac{1-e^k}{1+e^k} \overset{L}{=} \lim\limits_{k\to\infty} \dfrac{-e^k}{e^k} = -1$, therefore, the sequence converges.

c. $a_k = \left(1+\dfrac{1}{k}\right)^k$

Solution: $\lim\limits_{k\to\infty} \left(1+\dfrac{1}{k}\right)^k = \exp\left(\lim\limits_{k\to 0} \dfrac{\ln\left(1+\dfrac{1}{k}\right)}{\dfrac{1}{k}}\right) \overset{L}{=} \exp\left(\lim\limits_{k\to 0} \dfrac{\dfrac{1}{1+1/k}\cdot\dfrac{-1}{k^2}}{\dfrac{-1}{k^2}}\right) e^1$, therefore, the sequence converges

d. $a_k = \dfrac{k^3}{10k^2+1}$

Solution: $\lim\limits_{k\to\infty} \dfrac{k^3}{10k^2+1} = \infty$, therefore the sequence diverges.

2. Find the Taylor Series for

a. $f(x) = \dfrac{1}{(x-4)^2}$ at $x_0 = 5$.

Solution: $f(x) = (x-4)^{-2}$, $f'(x) = -2(x-4)^{-3}, \ldots, f^{(n)}(x) = (-1)^n (n+1)!(x-4)^{-(n+2)}$, so $f^{(n)}(5) = (-1)^n (n+1)!$ Thus

$$\frac{1}{(x-4)^2} = \sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!}(x-5)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{n!}(x-5)^n = \sum_{n=0}^{\infty} (-1)^n (n+1)(x-5)^n =$$

b. $f(x) = \sin x$ at $x = \dfrac{\pi}{2}$

Solution: $f(x) = \sin x,\ f'(x) = \cos x,\ f''(x) = -\sin x,\ f^{(3)}(x) = -\cos x,\ f^{(4)}(x) = \sin x, \ldots$ so $f\left(\dfrac{\pi}{2}\right) = 1,\ f'\left(\dfrac{\pi}{2}\right) = 0,\ f''\left(\dfrac{\pi}{2}\right) = -1,\ f^{(3)}\left(\dfrac{\pi}{2}\right) = 0,\ f^{(4)}\left(\dfrac{\pi}{2}\right) = 1, \ldots$ Thus

$$\sin x = 1 + 0\left(x-\frac{\pi}{2}\right) - \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{0}{3!}\left(x-\frac{\pi}{2}\right)^3 + \frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4 + \frac{0}{5!}\left(x-\frac{\pi}{2}\right)^5 - \frac{1}{6!}\left(x-\frac{\pi}{2}\right)^6 + \ldots$$

$$= 1 - \frac{1}{2!}\left(x-\frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x-\frac{\pi}{2}\right)^6 + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(x-\frac{\pi}{2}\right)^{2k}$$

3. Give that $e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}$ and $\sin x=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$, find the power series for each of the following:

a. $g(x)=x^2e^x$

Solution $g(x)=x^2e^x=x^2\sum_{k=0}^{\infty}\frac{x^k}{k!}=\sum_{k=0}^{\infty}\frac{x^{k+2}}{k!}$

b. $g'(x)$

Solution $g'(x)=x^2e^x+2xe^x=\frac{d}{dx}\left(\sum_{k=0}^{\infty}\frac{x^{k+2}}{k!}\right)=\sum_{k=0}^{\infty}\frac{(k+2)x^{k+1}}{k!}$

c. $\int\frac{\sin x}{x}dx$

Solution: $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$ thus $\frac{\sin x}{x}=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}$, so

$$\int\frac{\sin x}{x}dx=\int\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}dx=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}+C$$

d. $\int\frac{e^x-1}{x}dx$

Solution: $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$, so $e^x-1=\sum_{n=1}^{\infty}\frac{x^n}{n!}=x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ and $\frac{e^x-1}{x}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}$

Now

$$\int\frac{e^x-1}{x}dx=\int\sum_{n=1}^{\infty}\frac{x^{n-1}}{n!}dx=\sum_{n=1}^{\infty}\frac{x^n}{n\cdot n!}+C$$

4. Find the sum of the following series:

a. $\sum_{k=1}^{\infty}\left(\frac{e}{\pi}\right)^k$

Solution: This is a geometric series with $a=\frac{e}{\pi}$ and $r=\frac{e}{\pi}$ so $S=\frac{e/\pi}{1-e/\pi}=\frac{e}{\pi-e}$

b. $\sum_{n=2}^{\infty}\frac{2}{n^2-1}$

Solution: this is a telescoping series $\frac{2}{n^2-1}=\frac{1}{n-1}-\frac{1}{n+1}$, so

$$S_n=a_2+a_3+a_4+\cdots+a_{n-1}+a_n$$
$$=\left(1-\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n-2}-\frac{1}{n}\right)+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$$
$$=1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}$$

And $S=\lim_{n\to\infty}S_n=\frac{3}{2}$

c. $\displaystyle\sum_{k=1}^{\infty}\left[\left(\frac{7}{11}\right)^{k}-\left(\frac{3}{5}\right)^{k+1}\right]$

Solution: $\displaystyle\sum_{k=1}^{\infty}\left[\left(\frac{7}{11}\right)^{k}-\left(\frac{3}{5}\right)^{k+1}\right]=\sum_{k=1}^{\infty}\left(\frac{7}{11}\right)^{k}-\sum_{k=1}^{\infty}\left(\frac{3}{5}\right)^{k+1}=\frac{7/11}{1-7/11}-\frac{9/25}{1-3/5}=\frac{17}{20}$

5. Determine whether the following series are absolutely convergent, conditionally convergent, or divergent. . Justify your answers by citing relevant tests or reason.

a. $\displaystyle\sum_{k=2}^{\infty}\frac{(-1)^{k}\sqrt{k}}{\ln k}$

Solution: $\displaystyle\lim_{k\to\infty}\frac{\sqrt{k}}{\ln k}\overset{L}{=}\lim_{k\to\infty}\frac{\frac{1}{2\sqrt{k}}}{\frac{1}{k}}=\lim_{k\to\infty}\frac{k}{2\sqrt{k}}\overset{L}{=}\lim_{k\to\infty}\frac{\sqrt{k}}{2}=\infty$; So $\displaystyle\lim_{k\to 0}\frac{(-1)^{k}\sqrt{k}}{\ln k}\neq 0$ thus $\displaystyle\sum_{k=2}^{\infty}\frac{(-1)^{k}\sqrt{k}}{\ln k}$ diverges by the Divergence Test.

b. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k^{1/3}}$

Solution: $\displaystyle\sum_{k=1}^{\infty}|a_k|=\sum_{k=1}^{\infty}\frac{1}{k^{1/3}}$ is divergent since it is a p-series with $p=1/3<1$. However,

$\displaystyle\frac{1}{k^{1/3}}>0,\ \ \frac{1}{k^{1/3}}>\frac{1}{(k+1)^{1/3}}$, and $\displaystyle\lim_{k\to\infty}\frac{1}{k^{1/3}}=0$ thus by the Alternating Series Test, $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k^{1/3}}$ is convergent. Now since $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k^{1/3}}$ is convergent but $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^{1/3}}$ diverges so $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k^{1/3}}$ is conditionally convergent.

c. $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k^{3}}$

Solution: $\displaystyle\sum_{k=1}^{\infty}|a_k|=\sum_{k=1}^{\infty}\frac{1}{k^{3}}$ is convergent since it is a p-series, $p=3>1$ therefore $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k}}{k^{3}}$ is absolutely convergent.

6. Determine the interval of convergence for the following power series:

a. $\displaystyle\sum_{k=0}^{\infty}\frac{k!x^{2k}}{10^{k}}$

Solution: Use the ratio test to determine convergence of $\sum|a_k|$.

$\displaystyle\lim_{k\to\infty}\left|\frac{(k+1)!x^{2k+2}}{10^{k+1}}\cdot\frac{10^{k}}{k!x^{2k}}\right|=\frac{x^{2}}{10}\lim_{k\to\infty}(k+1)=\infty$ Thus $\displaystyle\sum_{k=0}^{\infty}\frac{k!x^{2k}}{10^{k}}$ converges only if $x=0$ otherwise it diverges.

b. $\displaystyle\sum_{k=1}^{\infty}\frac{(x-1)^{k}}{k3^{k}}$

Solution: Use the ratio test to determine convergence of $\sum|a_k|$.

$\lim_{k\to\infty}\left|\frac{(x-1)^{k+1}}{(k+1)3^{k+1}}\cdot\frac{k3^k}{(x-1)^k}\right| = \frac{|x-1|}{3}\lim_{k\to\infty}\frac{k}{k+1} = \frac{|x-1|}{3}$ thus the series converges absolutely if $\frac{|x-1|}{3} < 1 \Leftrightarrow |x-1| < 3 \Leftrightarrow -2 < x < 4$ Check the endpoints $x=-2$ and $x=4$ at $x=-2$ we get $\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ which converges conditionally it is the alternating harmonic series. At $x=4$ we get $\sum_{k=1}^{\infty}\frac{1}{k}$ which is a divergent p-series. Thus the IOC is $[-2,4)$

c. $\sum_{k=0}^{\infty}\frac{(2x-1)^k}{k^2+1}$

Solution: Use the ratio test to determine convergence of $\sum|a_k|$.

$\lim_{k\to\infty}\left|\frac{(2x-1)^{k+1}}{(k+1)^2+1}\cdot\frac{k^2+1}{(2x-1)^k}\right| = |2x-1|\lim_{k\to\infty}\frac{k^2+1}{(k+1)^2+1} = |2x-1|$ thus the series converges absolutely if $|2x-1| < 1 \Leftrightarrow 0 < x < 1$. Check the endpoints $x=0$ and $x=1$. At $x=1$, $\sum_{k=0}^{\infty}\frac{1}{k^2+1}$, compare to $\sum_{k=1}^{\infty}\frac{1}{k^2}$, now $\frac{1}{k^2+1} < \frac{1}{k^2}$ $\forall k \ge 1$ and since $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is a convergent p-series ($p=2$) then $\sum_{k=1}^{\infty}\frac{1}{k^2+1}$ converges (absolutely) by comparison test so $\sum_{k=0}^{\infty}\frac{1}{k^2+1}$ also converges. At $x=0$, $\sum_{k=0}^{\infty}\frac{(-1)^k}{k^2+1}$, consider $\sum|a_k| = \sum\frac{1}{k^2+1}$ which is convergent from above thus $\sum_{k=0}^{\infty}\frac{(-1)^k}{k^2+1}$ converges absolutely. Thus the IOC is $[0,1]$.

7. Determine whether the following series converge or diverge. Justify your answers by citing relevant tests or reason.

a. $\sum_{k=0}^{\infty}\frac{(-2)^k}{3^k+1}$

Solution: Look at $\sum|a_k| = \sum\frac{2^k}{3^k+1}$ this series looks like $\sum\frac{2^k}{3^k}$. Now $3^k+1 > 3^k \Leftrightarrow \frac{1}{3^k+1} < \frac{1}{3^k} \Leftrightarrow \frac{2^k}{3^k+1} < \frac{2^k}{3^k}$ $\forall k \ge 1$ and since $\sum\frac{2^k}{3^k}$ is a convergent geometric series, $r=\frac{2}{3}$, then $\sum_{k=0}^{\infty}\frac{(-2)^k}{3^k+1}$ is absolutely convergent by the CT which implies $\sum_{k=0}^{\infty}\frac{(-2)^k}{3^k+1}$ is convergent.

b. $\sum_{k=0}^{\infty}\frac{k!}{e^{k^2}}$,

Solution: $\lim_{k\to\infty}\left(\frac{(k+1)!}{e^{(k+1)^2}}\cdot\frac{e^{k^2}}{k!}\right) = \lim_{k\to\infty}\frac{k+1}{e^{2k+1}} = 0$ therefore $\sum_{k=0}^{\infty}\frac{k!}{e^{k^2}}$ is convergent by the Ratio Test.

c. $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$

Solution: Looks like $\sum \frac{\sqrt{n^2}}{n^3} = \sum \frac{1}{n^2}$. Now

$$\lim_{n\to\infty} \frac{\left(\frac{\sqrt{n^2-1}}{n^3+2n^2+5}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{n^2\sqrt{n^2-1}}{n^3+2n^2+5} = \lim_{n\to\infty} \frac{\sqrt{n^6-n^4}}{n^3+2n^2+5} = \lim_{n\to\infty} \frac{\sqrt{1-1/n^2}}{1+2/n+5/n^3} = 1$$ and since $\sum \frac{1}{n^2}$ is a convergent p-series (p=2) then by the LCT $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{n^3+2n^2+5}$ is convergent.

d. $\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}$

Solution: $\frac{1}{k \ln k} > 0 \ \forall k \geq 2$, $\frac{1}{(k+1)\ln(k+1)} > \frac{1}{k \ln k} \ \forall k \geq 2$, and $\lim_{k\to\infty} \frac{1}{k \ln k} = 0$ thus by the AST $\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln k}$ is convergent.

e. $\sum_{j=2}^{\infty} \frac{1}{j\sqrt{\ln j}}$

Solution: Let $a(x) = \frac{1}{x\sqrt{\ln x}}$ for $[2,\infty)$. Now $a(x)$ is a positive and continuous function. Also $a'(x) = \frac{-1(2\ln x+1)}{2x^2(\ln x)^{3/2}} < 0 \ \forall x \geq 2$, so $a(x)$ is a decreasing function.

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{N\to\infty} \int_2^{N} \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{N\to\infty} 2\sqrt{\ln x}\Big|_2^N = \lim_{N\to\infty}\left(2\sqrt{\ln N} - 2\sqrt{\ln 2}\right) = \infty$$

By the integral test since $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx$ diverges then $\sum_{j=2}^{\infty} \frac{1}{j\sqrt{\ln j}}$ diverges

f. $\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n$

Solution: $\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n}{8n+1}\right)^n} = \lim_{n\to\infty} \frac{3n}{8n+1} = \frac{3}{8} < 1$, Thus by the Root Test $\sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n$ converges.

g. $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k\sqrt{k}}$

Solution: Note $0 \leq \tan^{-1} k \leq \frac{\pi}{2} \ \forall k \geq 1$ which implies that $\frac{\tan^{-1} k}{k^{3/2}} < \frac{\pi/2}{k^{3/2}} \ \forall k \geq 1$ and since $\frac{\pi}{2}\sum \frac{1}{k^{3/2}}$ converges (it is a constant times a convergent p-series) then $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{k\sqrt{k}}$ is convergent by the CT.

h. $\sum_{n-1}^{\infty}(-1)^n \frac{n}{n^2+25}$

Solution: $\frac{n}{n^2+25} > 0 \ \forall \text{n} \geq 1$ and $f'(x) = \frac{25-x^2}{(x^2+25)^2} \leq 0 \ \forall x \geq 5$ so $u_n \geq u_{n+1} \ \forall n \geq 5$ thus by AST $\sum_{n=5}^{\infty}(-1)^n \frac{n}{n^2+25}$ is convergent and so $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+25}$ is also convergent since it is eventually the same.