1. Find the interval and radius of convergence for each of the following

a. $\displaystyle\sum_{k=0}^{\infty} \frac{10^k x^{2k}}{k!}$

Solution:

$$\rho = \lim_{k\to\infty} \left| \frac{10^{k+1} x^{2(k+1)}}{(k+1)!} \cdot \frac{k!}{10^k x^{2k}} \right| = \lim_{k\to\infty} \left| \frac{10^k \cdot 10}{10^k} \cdot \frac{k!}{(k+1)!} \cdot \frac{x^{2k} \cdot x^2}{x^{2k}} \right|$$
$$= 10x^2 \lim_{k\to\infty} \frac{1}{k+1} = 0 < 1$$

So the interval of convergence $-\infty < x < \infty$ and the radius is $R = \infty$.

b. $\displaystyle\sum_{k=1}^{\infty} \frac{(x-1)^k}{k3^k}$

Solution:

$$\rho = \lim_{k\to\infty} \left| \frac{(x-1)^{k+1}}{(k+1)3^{k+1}} \cdot \frac{k3^k}{(x-1)^k} \right| = \lim_{k\to\infty} \left| \frac{3^k}{3^k \cdot 3} \cdot \frac{k}{k+1} \cdot \frac{(x-1)^k \cdot (x-1)}{(x-1)^k} \right|$$
$$= \frac{|x-1|}{3} \lim_{k\to\infty} \frac{k}{k+1} = \frac{|x-1|}{3}$$

Thus

$\rho < 1 \iff \dfrac{|x-1|}{3} < 1 \iff |x-1| < 3 \iff -3 < x-1 < 3 \iff -2 < x < 4$

If $x = 4$ then $\displaystyle\sum_{k=1}^{\infty} \frac{(4-1)^k}{k3^k} = \sum_{k=1}^{\infty} \frac{1}{k}$ which is a divergent p-series, $p = 1$.

If $x = -2$ then $\displaystyle\sum_{k=1}^{\infty} \frac{(-2-1)^k}{k3^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ which is an alternating harmonic that converges.

So the interval of convergence $-2 \leq x < 4$ and the radius is $R = 3$.

c. $\displaystyle\sum_{k=0}^{\infty} \frac{(2x-1)^k}{k^2+1}$

Solution:

$$\rho = \lim_{k\to\infty} \left| \frac{(2x-1)^{k+1}}{(k+1)^2+1} \cdot \frac{k^2+1}{(2x-1)^k} \right| = \lim_{k\to\infty} \left| \frac{k^2+1}{k^2+2k+2} \cdot \frac{(2x-1)^k \cdot (2x-1)}{(2x-1)^k} \right|$$
$$= |2x-1| \lim_{k\to\infty} \frac{k^2}{k^2+2k+2} = |2x-1|$$

Thus

$\rho < 1 \iff |2x-1| < 1 \iff -1 < 2x-1 < 1 \iff 0 < 2x < 2 \iff 0 < x < 1$

If $x = 1$ then $\sum_{k=1}^{\infty} \frac{(2-1)^k}{k^2+1} = \sum_{k=1}^{\infty} \frac{1}{k^2+1}$ this series looks like $\sum_{k=1}^{\infty} \frac{1}{k^2}$ which is a convergent p-series, $p = 2$ and since $0 \leq \frac{1}{k^2+1} \leq \frac{1}{k^2}$ for all k then by Direct Comparison Test $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ is convergent.

If $x = 0$ then $\sum_{k=1}^{\infty} \frac{(0-1)^k}{k^2+1} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+1}$, since $\sum |a_k| = \sum_{k=1}^{\infty} \frac{1}{k^2+1}$ is convergent which implies $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2+1}$ is convergent (absolutely).

So the interval of convergence $0 \leq x \leq 1$ and the radius is $R = 1/2$.

2. Find the Taylor Series for each of the following

a. $f(x) = \sin x$ and $x = \pi/2$.

Solution:

k	$f^{(k)}(x)$	$f^k(\pi/2)$
0	$\sin x$	$\sin(\pi/2) = 1$
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1

And the Taylor Series is

$$f(x) = 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(x - \frac{\pi}{2}\right)^k$$

b. $f(x) = \frac{1}{(x-4)^2}$ for $x = 5$

Solution:

k	$f^{(k)}(x)$	$f^k(5)$
0	$(x-4)^{-2}$	$(1)^{-2}$
1	$-2(x-4)^3$	$-2(1)^{-3}$
2	$(2)(3)(x-4)^{-4}$	$3! \cdot (1)^{-4}$
3	$-(2)(3)(4)(x-4)^{-5}$	$-4! \cdot (1)^{-5}$
4	$(2)(3)(4)(5)(x-4)^{-6}$	$5! \cdot (1)^{-6}$

So $f^{(k)}(5) = (-1)^k(k+1)!$

And the Taylor Series is

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)!}{k!}(x-5)^k = \sum_{k=0}^{\infty} (-1)^k (k+1)(x-5)^k$$

c. $f(x) = \dfrac{1}{3x-2}$ at $x = 2$.

Solution:

k	$f^{(k)}(x)$	$f^k(2)$
0	$(3x-2)^{-1}$	$(4)^{-1}$
1	$-(3x-2)^{-2} \cdot 3$	$-(4)^{-2} \cdot 3$
2	$(1)(2)(3x-2)^{-3} \cdot 3 \cdot 3$	$2! \cdot (4)^{-3} \cdot 3^2$
3	$-(1)(2)(3)(3x-2)^{-4} \cdot 2^2 \cdot 2$	$-3! \cdot (4)^{-4} \cdot 3^3$
4	$(1)(2)(3)(4)(3x-2)^{-5} \cdot 2^3 \cdot 2$	$4! \cdot (4)^{-5} \cdot 3^4$

So $f^{(k)}(2) = (-1)^k k! 3^k 4^{-(k+1)} = \dfrac{(-1)^k k! 3^k}{4^{k+1}}$

And the Taylor Series is

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k k! 3^k}{4^{k+1}} \cdot \frac{1}{k!}(x-2)^k = \sum_{k=0}^{\infty} \frac{(-1)^k 3^k}{4^{k+1}}(x-2)^k$$

3. Given that $e^x = \sum_{k=0}^{\infty} \dfrac{x^k}{k!}$ and $\sin x = \sum_{k=0}^{\infty} \dfrac{(-1)^k x^{2k+1}}{(2k+1)!}$, find the power series for each of the following.

a. $g(x) = x^2 e^{3x}$

Solution:

$$g(x) = x^2 \sum_{k=0}^{\infty} \frac{(3x)^k}{k!} = \sum_{k=0}^{\infty} \frac{3^k x^{k+2}}{k!}$$

b. $g'(x)$

Solution:

$$g'(x) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{3^k x^{k+2}}{k!}\right) = \left(\sum_{k=0}^{\infty} \frac{3^k}{k!}\right) \frac{d(x^{k+2})}{dx} = \sum_{k=0}^{\infty} \frac{3^k}{k!} \left((k+2)x^{k+1}\right)$$

c. $\displaystyle\int \frac{\sin x}{x}\, dx$

Solution:

$$\frac{\sin x}{x} = \frac{1}{x} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!}$$

$$\int \frac{\sin x}{x}\,dx = \int \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!}\,dx$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int x^{2k}\,dx$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot \frac{x^{2k+1}}{2k+1} = C + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)(2k+1)!}$$

d. $\int \frac{e^x - 1}{x}\,dx$

Solution:

$$e^x - 1 = \sum_{k=0}^{\infty} \frac{x^k}{k!} - 1 = \sum_{k=1}^{\infty} \frac{x^k}{k!}$$

$$\frac{e^x - 1}{x} = \frac{1}{x} \sum_{k=1}^{\infty} \frac{x^k}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!}$$

$$\int \frac{e^x - 1}{x}\,dx = \int \sum_{k=1}^{\infty} \frac{x^{k-1}}{k!}\,dx = \sum_{k=1}^{\infty} \frac{1}{k!} \int x^{k-1}\,dx = C + \sum_{k=1}^{\infty} \frac{x^k}{(k)k!}$$

4. For the parametric curve $x = t^2 + 4$, $y = 2 - t$ for $-4 \le t \le 4$

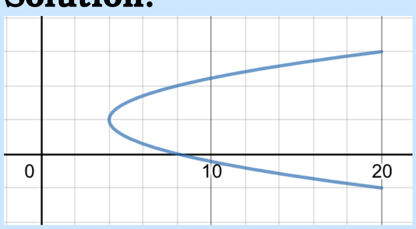
a. Eliminate the parameter to obtain an equation in x and y.

Solution:

$x = (2 - y)^2 + 4$ for $-2 \le y \le 6$

b. Graph the curve.

Solution:



5. Find an equation of the line tangent to the cycloid $x = t - \sin t$, $y = 1 - \cos t$ at $t = \pi/6$.

Solution:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{1 - \cos t}$$

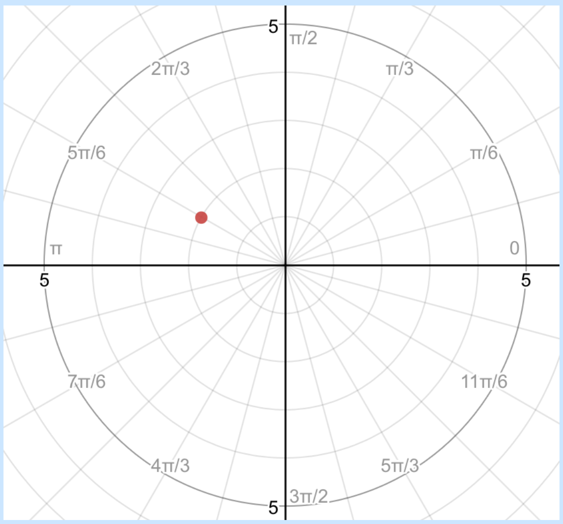
Thus the slope $m = \left.\frac{dy}{dx}\right|_{\pi/6} = \frac{\sin(\pi/6)}{1-\cos(\pi/6)} = \frac{1/2}{1-\sqrt{3}/2} = \frac{1}{2-\sqrt{3}}$,

$x_0 = \pi/6 - \sin(\pi/6) = \frac{\pi}{6} - \frac{1}{2}$, and $y_0 = 1 - \cos(\pi/6) = 1 - \frac{\sqrt{3}}{2}$

So the tangent line is $y - \left(1 - \frac{\sqrt{3}}{2}\right) = \frac{1}{2-\sqrt{3}}\left(x - \left(\frac{\pi}{2} - \frac{1}{2}\right)\right)$

6. Plot the point with polar coordinates $(2, 5\pi/6)$, then find the Cartesian coordinates of the points.

Solution:



Recall $x = r\cos\theta$ and $y = r\sin\theta$;

$x = 2\cos\left(\frac{5\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$ and $y = 2\sin\left(\frac{5\pi}{6}\right) = 2\left(\frac{1}{2}\right) = 1$

So $\left(2, \frac{5\pi}{6}\right) = \left(-\sqrt{3}, 1\right)$

7. For the point with Cartesian coordinate $(-\sqrt{3}, 3)$:

   a. Find the polar coordinates (r, θ) of the point where $r > 0$ and $0 \leq \theta < 2\pi$.

   Solution:
   Recall $r = \sqrt{x^2 + y^2}$ and $\tan(\theta) = \frac{y}{x}$

   $r = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$ and $\tan\theta = \frac{3}{-\sqrt{3}} = -\frac{\sqrt{3}}{1} = \frac{5\pi}{6}$

   $(-\sqrt{3}, 3) = \left(2\sqrt{3}, \frac{5\pi}{6}\right)$

b. Find the polar coordinates (r, θ) of the point where $r < 0$ and $0 \leq \theta < 2\pi$.

Solution:
Recall $r = \sqrt{x^2 + y^2}$ and $\tan(\theta) = \frac{y}{x}$

$r = \sqrt{3+9} = \sqrt{12} = 2\sqrt{3}$ and $\tan\theta = \frac{3}{-\sqrt{3}} = -\frac{\sqrt{3}}{1} = \frac{5\pi}{6}$

Also recall for $r < 0$, then $\theta = \theta \pm \pi$

$(-\sqrt{3}, 3) = \left(-2\sqrt{3}, \frac{11\pi}{6}\right)$

8. Replace the Cartesian equation by the equivalent polar equation.

a. $x + y = 4$

Solution:
Recall $x = r\cos\theta$ and $y = r\sin\theta$;

$x+y = 4 \iff r\cos\theta + r\sin\theta = 4 \iff r(\cos\theta + \sin\theta) = 4 \iff 4 = \frac{4}{\cos\theta + \sin\theta}$

b. $(x-5)^2 + y^2 = 25$

Solution:
$(x-5)^2 + y^2 = 25 \iff x^2 - 10x + 25 + y^2 = 25 \iff x^2 + y^2 = 10x$

Recall $r^2 = x^2 + y^2$ and $\tan(\theta) = \frac{y}{x}$

Thus $x^2 + y^2 = 10x \iff r^2 = 10r\cos\theta \iff r = 10\cos\theta$

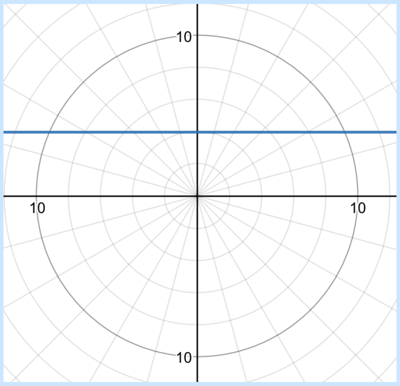
9. Replace the polar equation by the equivalent Cartesian equation. Then describe or identify each.

a. $r = 4\csc\theta$

Solution:
$r = 4\csc\theta = \frac{4}{\sin\theta} \iff r\sin\theta = 4$

Recall $x = r\cos\theta$ and $y = r\sin\theta$;

So $r\sin\theta = 4 \iff y = 4$ which is a horizontal line at 4.

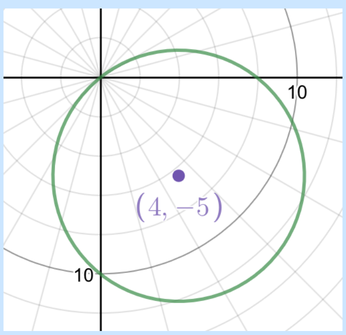


b. $r = 8\cos\theta - 10\sin\theta$

Solution:
$r = 8\cos\theta - 10\sin\theta \iff r^2 = 8r\cos\theta - 10r\sin\theta \iff x^2 + y^2 = 8x - 10y$

Recall $x = r\cos\theta$, $y = r\sin\theta$, and $r^2 = x^2 + y^2$;

Thus $x^2 + y^2 = 8x - 10y \iff x^2 - 8x + y^2 + 10y = 0 \iff x^2 - 8x + 16 - 16 + y^2 + 10y + 25 - 25 = 0 iff (x-4)^2 + (y+5)^2 = 41$, which is a circle centered at $(4, -5)$ with radius $\sqrt{41}$.



10. Write the equation of the tangent line to the curve $r = 1 + \sin\theta$ at $\theta = \dfrac{3\pi}{4}$.

Solution:
Recall $x = r\cos\theta$, $y = r\sin\theta$;

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta}$$

Thus the slope

$$
\begin{aligned}
m &= \left.\frac{dy}{dx}\right|_{3\pi/4} \\
&= \frac{-\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}\right) + \left(1+\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)}{-\frac{\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}\right) - \left(1+\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)} \\
&= \frac{-1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{-2-\sqrt{2}}{-\sqrt{2}} = \sqrt{2}+1
\end{aligned}
$$

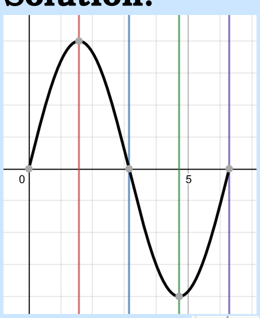
$x_0 = \left(1+\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}}{2} - \frac{1}{2}$, and $y_0 = \left(1+\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2} + \frac{1}{2}$ - So the tangent line is

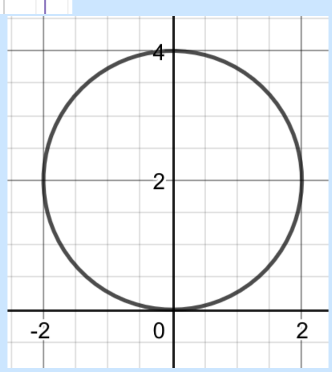
$$
y - \left(\frac{\sqrt{2}}{2} + \frac{1}{2}\right) = \left(\sqrt{2}+1\right)\left(x - \left(\frac{-\sqrt{2}}{2} - \frac{1}{2}\right)\right) \iff y = \left(\sqrt{2}+1\right)\left(x+\sqrt{2}+1\right)
$$

11. Graph the polar equation

a. $r = 4\sin\theta$

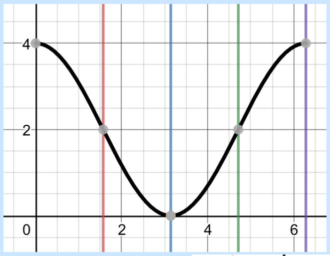
Solution:

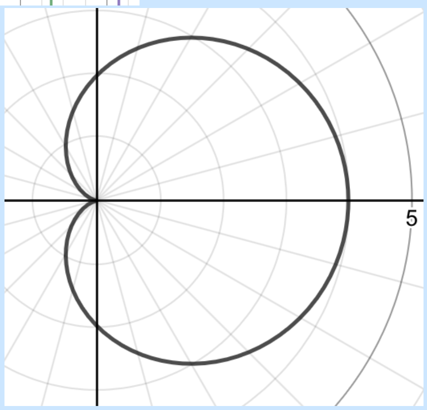


which gives 

b. $r = 2 + 2\cos\theta$

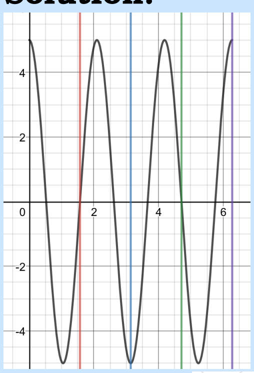
Solution:

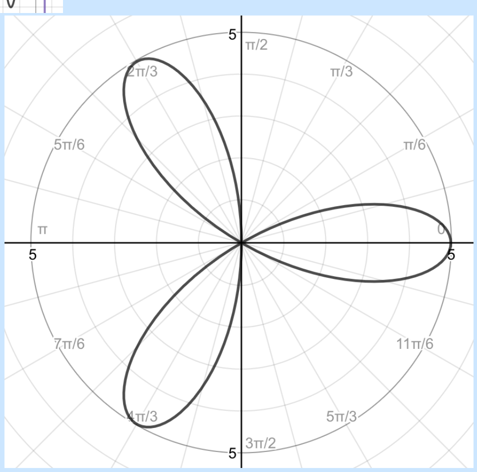


which gives 

c. $r = 5\cos 3\theta$

Solution:

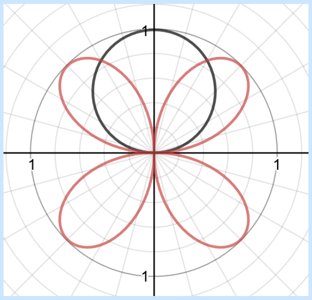


which gives 

12. Find the area that lies inside both curves $r = \sin 2\theta$ and $r = \sin\theta$.

Solution:

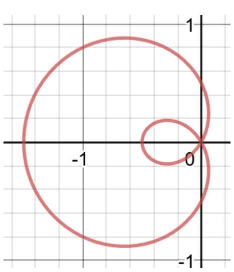
These curves intersect at $\sin 2\theta = \sin\theta \iff 2\sin\theta\cos\theta = \sin\theta \iff 2\sin\theta\cos\theta - \sin\theta = \sin\theta(2\cos\theta - 1) = 0$ which is true if $\sin\theta = 0$ or $\cos\theta = 1/2$ thus $\theta = 0, \pi, \pm\pi/3$



Area is

$$\begin{aligned} A &= 2\cdot\frac{1}{2}\left(\int_0^{\pi/3}\sin^2\theta\,d\theta + \int_{\pi/3}^{\pi/2}\sin^2 2\theta\,d\theta\right) \\ &= 2\cdot\frac{1}{2}\left(\int_0^{\pi/3}\frac{1}{2}(1-\cos 2\theta)\,\theta\,d\theta + \int_{\pi/3}^{\pi/2}\frac{1}{2}(1-\cos 4\theta)\,d\theta\right) \\ &= \frac{1}{2}\left[\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/3} + \left(\theta - \frac{1}{4}\sin 4\theta\right)\Big|_{\pi/3}^{\pi/2}\right] \\ &= \frac{1}{2}\left[\frac{\pi}{3} - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{2} - \frac{\pi}{3} + \frac{1}{4}\left(-\frac{\sqrt{3}}{2}\right)\right] \\ &= \frac{1}{2}\left[\frac{\pi}{2} - \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{8}\right] = \frac{\pi}{4} - \frac{3\sqrt{3}}{16} \end{aligned}$$

13. Find the area of the region enclosed by the inner loop of $r = \frac{1}{2} - \cos\theta$. Set up the integral but do not evaluate.



Solution:

$r = \frac{1}{2} - \cos\theta = 0 \iff \cos\theta = \frac{1}{2}$ this occurs for $\theta = \pm\pi/3$.

Note: $r(0) = \frac{1}{2} - \cos 0 = -\frac{1}{2}$

Area is

$$A = 2\left(\frac{1}{2}\right)\int_0^{\pi/3}\left(\frac{1}{2} - \cos\theta\right)^2 d\theta$$