Exam 4 **Name**______________________________
Math 132-01 **Calculus II** **August 2, 2019**

Guidelines

- **Calculators are not allowed.**
- Read the questions carefully. You have 65 minutes; use your time wisely.
- You may leave your answers in symbolic form, like $\sqrt{3}$ or $\ln(2)$, unless they simplify further like $\sqrt{9} = 3$ or $\cos(3\pi/4) = -\sqrt{2}/2$.
- Put a box around your final answers when relevant.
- Show all steps in your solutions and make your reasoning clear. Answers with no explanation will not receive full credit, even when correct.
- Use the space provided. If necessary, write "see other side" and continue working on the back of the same page.

---

1. (8 points) To be completed once exams are graded and returned. Please correct any problem with points deducted. All corrections should be completed neatly on a separate sheet of paper. Once you have finished your corrections, take your exam and corrections to the Office of Student Learning (OSL), and a tutor will check your answers and sign below. The checked solutions should be given to your instructor.

   Signature: ____________________________
   Print Name: ____________________________
   Date: ______________

2. For the point with Cartesian coordinates $\left(-3\sqrt{3}, -3\right)$, find polar coordinates (r, θ) with

   a. (3 points) $r > 0$ and $0 \le \theta < 2\pi$;

   Solution:
   $r = \sqrt{27+9} = \sqrt{36} = 6$ and $\tan\theta = \dfrac{-3}{-3\sqrt{3}} = \dfrac{1}{\sqrt{3}}$ so $\theta = \dfrac{7\pi}{6}$

   So the point is $\left(6, \dfrac{7\pi}{6}\right)$.

   b. (3 points) $r < 0$ and $0 \le \theta < 2\pi$.

   Solution:
   For a negative r, we need the angle to be in the quadrant I, opposite quadrant III where the point is.
   $\theta = \dfrac{7\pi}{6} \pm \pi = \dfrac{\pi}{6}$
   So the point is $\left(-6, \dfrac{\pi}{6}\right)$.

3. (8 points) Convert the Cartesian equation $(x-3)^2 + (y-2)^2 = 13$ to the equivalent polar equation.

Solution:
Rewriting the above equation and completing the square we get $(x-3)^2 + (y-2)^2 = 13 \iff x^2 - 6x + 9 + y^2 - 4y + 4 = 13 \iff x^2 - 6x + y^2 - 4y = 0 \iff x^2 + y^2 - 6x - 4y = 0$

Substitute into the equation above using $r^2 = x^2 + y^2$, $x = 4\cos\theta$, and $y = r\sin\theta$. So $x^2 + y^2 - 6x - 4y = 0 \iff r^2 - 6r\cos\theta - 4\sin\theta = 0 \iff r^2 = 6r\cos\theta + 4r\sin\theta = r(6\cos\theta + 4\sin\theta) \iff r = 6\cos\theta + 4\sin\theta$.

4. (8 points) Convert the polar equation $r = \dfrac{6}{\cos\theta - \sin\theta}$ to equivalent Cartesian equation.

Solution:
Remember $x = r\cos\theta$ and $y = r\sin\theta$ so,
$r = \dfrac{6}{\cos\theta - \sin\theta} \iff r(\cos\theta - \sin\theta) = 6 \iff r\cos\theta - r\sin\theta = 6 \iff x - y = 6$ or $y = x - 6$

5. Consider the parametric equations $x = \frac{1}{2}t^2$, $y = t^4$ for $-1 \leq t \leq \sqrt{4}$.

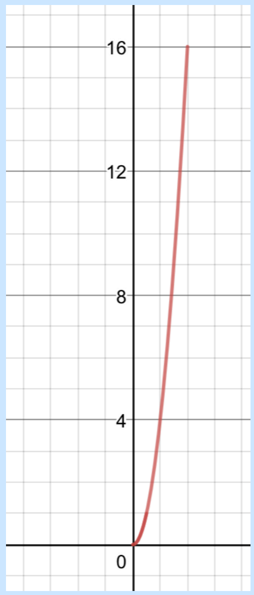
a. (6 points) Eliminate the parameter to obtain an equation in x and y.

Solution:

$x = \frac{1}{2}t^2 \iff 2x = t^2$, so $y = t^4 = (t^2)^2 = (2x)^2 = 4x^2$ for $0 \leq x \leq 2$

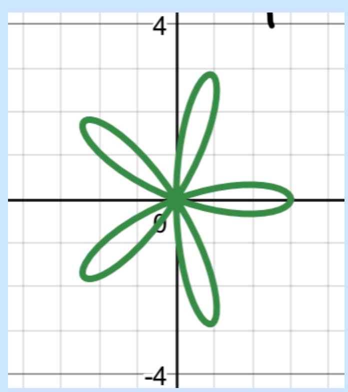
b. (4 points) Sketch the curve.

Solution:



6. (12 points) Sketch the loops of $r = 3\cos(5\theta)$ and find the area enclosed by one loop.

Solution:

Graph r with Cartesian	Graph Polar
	

Area

$$
\begin{aligned}
A = \frac{1}{2}\int_{-\pi/10}^{\pi/10} 9\cos^2(5\theta)\,d\theta &= \frac{9}{2}\cdot\frac{1}{2}\int_{-\pi/10}^{\pi/10} \left(1+\cos(10\theta)\right)\,d\theta \\
&= \frac{9}{4}\left(\theta + \frac{1}{10}\sin(10\theta)\right)\Bigg|_{-\pi/10}^{\pi/10} \\
&= \frac{9}{4}\left(\left(\frac{\pi}{10}+\frac{\pi}{10}\right) + \frac{1}{10}\left(\sin(\pi)-\sin(-\pi)\right)\right) \\
&= \frac{9}{4}\cdot\frac{\pi}{5} = \frac{9\pi}{20}.
\end{aligned}
$$

7. (8 points) Find an equation of the line tangent to the parametric curve $x = \cos(2t)$, $y = \sin(3t)$ at $t = -\dfrac{\pi}{12}$

Solution:

To find the slope first find $\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3\cos(3t)}{-2\sin(2t)}$:

so the slope $m = \left.\dfrac{dy}{dx}\right|_{t=-\pi/12} = \dfrac{3\cos(\pi/4)}{-2\sin(-\pi/6)} = \dfrac{3\cdot\sqrt{2}/2}{-2\cdot -1/2} = \dfrac{3\sqrt{2}}{2}$ and $x_0 = \cos(-\pi/6) = \dfrac{\sqrt{3}}{2}$ and $y_0 = \sin(-\pi/4) = -\dfrac{\sqrt{2}}{2}$

Thus the tangent line is $y + \dfrac{\sqrt{2}}{2} = \dfrac{3\sqrt{2}}{2}\left(x - \dfrac{\sqrt{3}}{2}\right)$

8. (8 points) Find the slope of the tangent line to the polar curve $r = 8\sin\theta$ at $\theta = \dfrac{5\pi}{6}$.

Solution:

Note: $f'(\theta) = 8\cos\theta$.

To find the slope first find $\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta} = \dfrac{8\cos\theta\sin\theta + 8\sin\theta\cos\theta}{8\cos\theta\cos\theta - 8\sin\theta\sin\theta}$:

$\left.\dfrac{dy}{dx}\right|_{\theta=5\pi/6} = \dfrac{8\cdot -\sqrt{3}/2\cdot 1/2 + 8\cdot 1/2\cdot -\sqrt{3}/2}{8\cdot -\sqrt{3}/2\cdot -\sqrt{3}/2 - 8\cdot 1/2\cdot 1/2} = \dfrac{-4\sqrt{3}}{4} = -\sqrt{3}$

9. (12 points) Determine the radius and interval of convergence for the series $\displaystyle\sum_{k=1}^{\infty} \frac{(x+3)^k}{k\,5^k}$

Solution:

$\displaystyle\lim_{k\to\infty}\left|\frac{(x+3)^{k+1}}{(k+1)5^{k+1}}\cdot\frac{k5^k}{(x+3)^k}\right| = \frac{|x+3|}{5}\lim_{k\to\infty}\frac{k}{k+1} = \frac{|x+3|}{5} < 1$

So the series converges if $-1 < \dfrac{x+3}{5} < 1 \iff -5 < x+3 < 5 \iff -8 < x < 2$

Check the endpoints:

if $x = 2$ $\sum \frac{5^k}{k \cdot 5^k} = \sum \frac{1}{k}$ which is a divergent p-series since $p = 1$.

if $x = -8$ $\sum \frac{(-5)^k}{k \cdot 5^k} = \sum \frac{(-1)^k}{k}$ which is the alternating harmonic series that converges.

So the interval of convergence $-8 \leq x < 2$ and the radius is 5.

10. (12 points) Find the Taylor Series for $f(x) = \dfrac{1}{3x-2}$ at $a = 2$.

Solution:

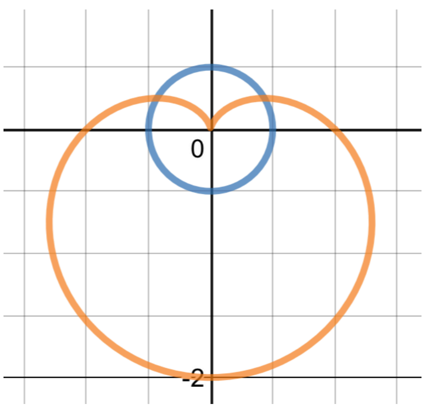
k	$f^{(k)}(x)$	$f^k(2)$
0	$(3x-2)^{-1}$	$(4)^{-1}$
1	$-(3x-2)^{-2} \cdot 3$	$-(4)^{-2} \cdot 3$
2	$(1)(2)(3x-2)^{-3} \cdot 3 \cdot 3$	$2! \cdot (4)^{-3} \cdot 3^2$
3	$-(1)(2)(3)(2x+2)^{-4} \cdot 2^2 \cdot 2$	$-3! \cdot (4)^{-4} \cdot 3^3$
4	$(1)(2)(3)(4)(2x+2)^{-5} \cdot 2^3 \cdot 2$	$4! \cdot (4)^{-5} \cdot 3^4$

So $f^{(k)}(2) = (-1)^k k! 3^k 4^{-(k+1)} = \dfrac{(-1)^k k! 3^k}{4^{k+1}}$

And the Taylor Series is

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k k! 3^k}{4^{k+1}} \cdot \frac{1}{k!}(x-2)^k = \sum_{k=0}^{\infty} \frac{(-1)^k 3^k}{4^{k+1}}(x-2)^k$$

11. (8 points) Find the area inside the circle $r = \frac{1}{2}$ and outside $r = 1 - \sin\theta$. Set up the integral; but, do not evaluate.



Solution:

Points of Intersection: $1 - \sin\theta = \dfrac{1}{2} \iff \sin\theta = \dfrac{1}{2}$ so $\theta = \dfrac{\pi}{6}$ and $\theta = \dfrac{5\pi}{6}$

Area

$$A = 2\left[\frac{1}{2}\int_{\pi/6}^{\pi/2} \left(\frac{1}{2}\right)^2 d\theta - \frac{1}{2}\int_{\pi/6}^{\pi/2} (1 - \sin\theta)^2 \, d\theta\right]$$

Question	Points	Score
1	8	
2	6	
3	8	
4	8	
5	10	
6	12	
7	8	
8	8	
9	12	
10	12	
11	8	
Total:	100	