Exam 3 **Name**______________________________

Math 132-01 **Calculus II** **July 22, 2019**

Guidelines

- **Calculators are not allowed.**
- Read the questions carefully. You have 65 minutes; use your time wisely.
- You may leave your answers in symbolic form, like $\sqrt{3}$ or $\ln(2)$, unless they simplify further like $\sqrt{9} = 3$ or $\cos(3\pi/4) = -\sqrt{2}/2$.
- Put a box around your final answers when relevant.
- Show all steps in your solutions and make your reasoning clear. Answers with no explanation will not receive full credit, even when correct.
- Use the space provided. If necessary, write ”see other side” and continue working on the back of the same page.

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1. (8 points) To be completed once exams are graded and returned. Please correct any problem with points deducted. All corrections should be completed neatly on a separate sheet of paper. Once you have finished your corrections, take your exam and corrections to the Office of Student Learning (OSL), and a tutor will check your answers and sign below. The checked solutions should be given to your instructor.

   Signature: ______________________________
   Print Name: ______________________________
   Date: _______________

Tests	Conditions	Conclusion
Positive Term Tests		
Integral Test $a_k = a(k)$, $a(x)$ is positive, continuous, decreasing	$\int_1^\infty a(x)dx$ converges	Converges
	$\int_1^\infty a(x)dx$ diverges	Diverges
Comparison Test	$0 \le a_k \le b_k$ and $\sum_{k=1}^{\infty} b_k$ converges	Converges
	$0 \le b_k \le a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges	Diverges
Limit Comparison Test $\lim_{k\to\infty} \frac{a_k}{b_k} = L$, $0 < L < \infty$	$\sum_{k=1}^{\infty} b_k$ converges	Converges
	$\sum_{k=1}^{\infty} b_k$ diverges	Diverges
Ratio Test $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \rho$	$\rho < 1$	Converges
	$\rho > 1$	Diverges
	$\rho = 1$	Inconclusive
Root Test $\lim_{k\to\infty} [a_k]^{1/k} = \rho$	$\rho < 1$	Converges
	$\rho > 1$	Diverges
	$\rho = 1$	Inconclusive
Series with some non-positive terms		
Alternating Series Test $a_k > 0$ and $0 < a_{k+1} \le a_k$	$\lim_{k\to\infty} a_k = 0$	Converges
	$\lim_{k\to\infty} a_k \neq 0$	Use Divergence Test to show divergent

Question	Points	Score
1	8	
2	6	
3	6	
4	6	
5	6	
6	8	
7	6	
8	10	
9	8	
10	8	
11	8	
12	8	
13	12	
Total:	100	

2. (6 points) Determine whether the sequence $\left\{\dfrac{k^2+7}{\sqrt{9k^4+1}}\right\}$ is convergent or divergent.

Solution:
$\lim_{k\to\infty} \dfrac{k^2+7}{\sqrt{9k^4+1}} = \dfrac{1}{3}$ sequence converges

3. (6 points) Determine if the series $\sum_{k=1}^{\infty} k^{-4/5}$ is convergent or divergent. Explain your reasoning.

Solution:
The series is a p-series with $p = 4/5 < 1$ therefore it diverge

4. (6 points) Determine if the series $\sum_{k=1}^{\infty} \dfrac{k^3}{k^3+1}$ is convergent or divergent. Explain your reasoning.

Solution:
$\lim_{k\to\infty} \dfrac{k^3}{k^3+1} = 1 \neq 0$, therefore it diverges by the Divergence Test.

5. (6 points) Determine if the series $\dfrac{1}{16} + \dfrac{3}{64} + \dfrac{9}{256} + \dfrac{27}{1024} + \cdots$ is convergent or divergent. Explain your reasoning.

Solution:
$\sum_{k=0}^{\infty} \dfrac{3^k}{16 \cdot 4^k}$ geometric series with $r = \dfrac{3}{4}$ therefore it converges.

6. (8 points) Use the integral test to determine if the series $\sum_{k=2}^{\infty} \frac{1}{k\sqrt[3]{\ln k}}$ is convergent or divergent (be sure to show that three conditions apply).

Solution:
$f(x) = \frac{1}{x\sqrt[3]{\ln x}}$ is positive and continuous $\forall k \geq 1$ and $f'(x) = -\frac{3\ln x + 1}{x^2(\ln x)^{4/3}} < 0$ if $x \geq 2$ so f decreasing. $\int_2^{\infty} \frac{1}{x\sqrt[3]{\ln x}}dx = \lim_{B\to\infty} \left(\frac{2}{3}(\ln B)^{2/3} - \frac{2}{3}(\ln 2)^{2/3}\right) = \infty$ Thus the integral diverges so by the integral test, the series also diverges.

7. (6 points) Find the sum of the series $\sum_{k=3}^{\infty} \frac{2}{3^k}$.

Solution:
For the geometric series: $c = \frac{2}{27}$ and $r = \frac{1}{3}$ so $S = \frac{2/27}{1 - 1/3} = \frac{1}{9}$

8. (10 points) Determine if the series $\sum_{k=1}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+3}\right)$ is convergent or divergent. If it converges, what is the sum?

Solution:
$S_n = a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n+1}\right) + \left(\frac{1}{n} - \frac{1}{n+2}\right) + \left(\frac{1}{n+1} - \frac{1}{n+3}\right) = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$ and $\lim_{k\to\infty} S_n = \frac{5}{6}$ therefore the series converges and the sum is $\frac{5}{6}$

9. (8 points) Determine if the series $\sum_{k=1}^{\infty} \frac{\cos k}{k^3}$ converges or diverges. Explain your reasoning.

Solution:

It is known that $0 \leq |\cos k| \leq 1 \; \forall k$ and since $k^3 > 0 \; \forall k > 1$ then $0 \leq \frac{|\cos k|}{k^3} \leq \frac{1}{k^3}$ and $\sum \frac{1}{k^3}$ is a p-series, $p = 3 > 1$ so it is convergent. Thus by the Comparison Test $\sum \frac{|\cos k|}{k^3}$ is absolutely convergent which implies the series converges.

10. (8 points) Determine if the series $\sum_{k=1}^{\infty} \frac{2k+1}{4^k}$ converges or diverges. Explain your reasoning.

Solution:

$\rho = \lim_{k\to\infty} \left(\frac{2k+3}{4^{k+1}} \cdot \frac{4^k}{2k+1} \right) = \frac{1}{4} \lim_{k\to\infty} \frac{2k+3}{2k+1} = \frac{1}{4} < 1$

Thus by the Ratio Test, the series converges.

11. (8 points) Determine if the series $\sum_{k=1}^{\infty} \left(\frac{2k^2}{3k^2+1} \right)^k$ converges or diverges. Explain your reasoning.

Solution:

$\lim_{k\to\infty} \left[\left(\frac{2k^2}{3k^2+1} \right)^k \right]^{1/k} = \frac{2}{3} < 1$. Thus the series converges by the Root Test.

12. (8 points) Determine if the series $\sum_{k=1}^{\infty} \frac{k^2}{k^4+k^2+2}$ converges or diverges. Explain your reasoning.

Solution:

The series looks like $\sum b_k = \sum_{k=1}^{\infty} \frac{k^2}{k^4} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ this is a convergent p-series $p = 2 > 1$.

$\lim_{k\to\infty} \left(\frac{k^2}{k^4+k^2+2} \cdot \frac{k^2}{1} \right) = 1$ and $0 < 1 < \infty$ so by the Limit Comparison Test the series $\sum_{k=1}^{\infty} \frac{k^2}{k^4+k^2+2}$ also is convergent.

OR

Since $k^4 + k^2 + 2 > k^4$ for all $k \geq 1$ which implies that $0 < \dfrac{1}{k^4 + k^2 + 2} < \dfrac{1}{k^4}$ and since $k^2 > 0$ for all $k \geq 1$ then $0 < \dfrac{k^2}{k^4 + k^2 + 2} < \dfrac{k^2}{k^4} = \dfrac{1}{k^2}$ so by the Direct Comparison Test, the series $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{k^4 + k^2 + 2}$ also converges.

13. Given the series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{k^3+32}$,

a. (6 points) Show that the series converges. Show all your work, explain your answer.

Solution:
$\displaystyle\lim_{k\to\infty} \frac{k^2}{k^3+32} = 0$ and $a'(x) = \dfrac{2x(x^3+32) - x^2(3x^2)}{(x^2+16)^2} = \dfrac{x(64-x^3)}{(x^3+32)^2} < 0,\ \forall x > 4$ so a_k is decreasing $\forall k > 4$
Thus by the AST the series converges.

b. (6 points) Determine whether the series converges absolutely or conditionally.

Solution:
$\sum |a_k| = \sum \dfrac{k^2}{k^3+32}$ which looks like $\sum \dfrac{1}{k}$ this is a divergent p-series $p = 1$. $\displaystyle\lim_{k\to\infty} \left(\frac{k^2}{k^3+32} \cdot \frac{k}{1}\right) = \lim_{k\to\infty} \frac{k^3}{k^3+32} = 1$ and since $0 < 1 < \infty$. By the limit comparison test, the series diverges.
Therefore $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k^2}{k^3+32}$ is conditionally convergent.