Exam 3 **Name: Solution**
Math 132-02 **Calculus II** **November 7, 2018**

Guidelines

- **Calculators are not allowed.**
- Read the questions carefully. You have 65 minutes; use your time wisely.
- You may leave your answers in symbolic form, like $\sqrt{3}$ or $\ln(2)$, unless they simplify further like $\sqrt{9} = 3$ or $\cos(3\pi/4) = -\sqrt{2}/2$.
- Put a box around your final answers when relevant.
- Show all steps in your solutions and make your reasoning clear. Answers with no explanation will not receive full credit, even when correct.
- Use the space provided. If necessary, write "see other side" and continue working on the back of the same page.

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1. (8 points) To be completed once exams are graded and returned. Please correct any problem with points deducted. All corrections should be completed neatly on a separate sheet of paper. Once you have finished your corrections, take your exam and corrections to the Office of Student Learning (OSL), and a tutor will check your answers and sign below. The checked solutions should be given to your instructor.

   Signature: ______________________________
   Print Name: ______________________________
   Date: _______________

Tests	Conditions	Conclusion
Positive Term Tests		
Integral Test $a_k = a(k)$, $a(x)$ is positive, continuous, decreasing	$\int_1^\infty a(x)dx$ converges	Converges
	$\int_1^\infty a(x)dx$ diverges	Diverges
Comparison Test	$0 \leq a_k \leq b_k$ and $\sum_{k=1}^{\infty} b_k$ converges	Converges
	$0 \leq b_k \leq a_k$ and $\sum_{k=1}^{\infty} b_k$ diverges	Diverges
Limit Comparison Test $\lim_{k\to\infty} \frac{a_k}{b_k} = L$, $0 < L < \infty$	$\sum_{k=1}^{\infty} b_k$ converges	Converges
	$\sum_{k=1}^{\infty} b_k$ diverges	Diverges
Ratio Test $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \rho$	$\rho < 1$	Converges
	$\rho > 1$	Diverges
	$\rho = 1$	Inconclusive
Root Test $\lim_{k\to\infty} [a_k]^{1/k} = \rho$	$\rho < 1$	Converges
	$\rho > 1$	Diverges
	$\rho = 1$	Inconclusive
Series with some non-positive terms		
Alternating Series Test $a_k > 0$ and $0 < a_{k+1} \leq a_k$	$\lim_{k\to\infty} a_k = 0$	Converges
	$\lim_{k\to\infty} a_k \neq 0$	Use Divergence Test to show divergent

Question	Points	Score
1	8	
2	12	
3	8	
4	8	
5	8	
6	8	
7	8	
8	16	
9	12	
10	12	
Total:	100	

2. Determine whether the following sequences converge or diverge. If it is convergent, find its limit.

   a. (6 points) $\left\{ \dfrac{3+5k^2}{k+k^2} \right\}$

   Solution:
   $\lim\limits_{k\to\infty} \dfrac{3+5k^2}{k+k^2} = 5$ thus the sequence converges (remember that for a sequence to converge, the limit just has to exist).

   b. (6 points) $\left\{ \dfrac{(2n-1)!}{(2n+1)!} \right\}$

   Solution:
   $\lim\limits_{k\to\infty} \dfrac{(2n-1)!}{(2n+1)!} = \lim\limits_{k\to\infty} \dfrac{(2n-1)!}{(2n+1)(2n)(2n-1)!} = \lim\limits_{k\to\infty} \dfrac{1}{(2n+1)(2n)} = 0$ thus the sequence converges.

3. (8 points) Find the sum of the series $\displaystyle\sum_{k=0}^{\infty} \frac{3^k}{\pi^{k+1}}$.

   Solution:
   This is a geometric series $\displaystyle\sum_{k=0}^{\infty} \frac{3^k}{\pi^{k+1}} = \frac{1}{\pi} + \frac{3}{\pi^2} + \frac{3^2}{\pi^3} + \cdots = \frac{1/\pi}{1-3/\pi} = \frac{1}{\pi-3}$

   The formula is $S = \dfrac{a}{1-r}$, a is the first term of the series and r is the multiplier.

4. (8 points) Determine if the series $\displaystyle\sum_{k=1}^{\infty} \frac{k^{2k}}{(1+2k^2)^k}$ converges or diverges. Explain your reasoning.

   Solution:
   $\displaystyle\lim_{k\to\infty} \left[\frac{k^{2k}}{(1+2k^2)^k} \right]^{1/k} = \lim_{k\to infty} \frac{k^2}{1+2k^2} = \frac{1}{2} < 1$
   Thus the series converges by the root test.

5. (8 points) Determine if the series $\sum_{k=1}^{\infty} \frac{\sin(4k)}{4^k}$ converges or diverges. Explain your reasoning.

Solution:
This is NOT an alternating series and all the terms are not positive so consider the series $\sum_{k=1}^{\infty} \left| \frac{\sin(4k)}{4^k} \right| = \sum_{k=1}^{\infty} \frac{|\sin(4k)|}{4^k}$

Now $0 \leq |\sin(4k)| \leq 1$ for all $k \geq 1$ which implies that $0 \leq \frac{|\sin(4k)|}{4^k} \leq \frac{1}{4^k}$ for all $k \geq 1$. We know that $\sum_{k=1}^{\infty} \frac{1}{4^k}$ is a geometric series with $r = \frac{1}{4}$ so $|r| < 1$ and it converges.

Thus by the comparison test $\sum_{k=1}^{\infty} \frac{|\sin(4k)|}{4^k}$ converges which implies that $\sum_{k=1}^{\infty} \frac{\sin(4k)}{4^k}$ also converges.

6. (8 points) Determine if the series $\sum_{k=1}^{\infty} \frac{(k+1)^2}{k(k+2)}$ converges or diverges. Explain your reasoning.

Solution:
$\lim_{k\to\infty} \frac{(k+1)^2}{k(k+2)} = 1 \neq 0$, therefore it diverges by the Divergence Test.

7. (8 points) Determine if the series $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ converges or diverges. Explain your reasoning.

Solution:

$f(x) = \frac{1}{x \ln x}$ is a positive and continuous function for $x \geq 2$ because it is a combination of positive and continuous functions. Also $f'(x) = -\frac{\ln x + 1}{(x \ln x)^2} < 0$ for all $x \geq 2$, f is decreasing.

$\int_2^n \infty \frac{1}{x \ln x}\, dx = \lim_{B\to\infty} \ln|\ln x|\Big|_2^B = \lim_{B\to\infty} (\ln|\ln B| - \ln|\ln 2|) = \infty$, thus the integral diverges.

So by the Integral Test, the series also diverges.

8. a. (8 points) Show the series $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^2+100}$ converges. Explain your reasoning.

Solution:
$a_k = \frac{k}{k^2+100}$ and $\lim_{k\to\infty} \frac{k}{k^2+100} = 0$

Also $f'(x) = \frac{1(x^2+100) - x(2x)}{(x^2+100)^2} = \frac{100-x^2}{(x^2+100)^2} \leq 0$ for $k \geq 10$ so the sequence of terms in non-increasing.

So by the Alternating Series Test, the series converges.

b. (8 points) Determine if the series $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^2+100}$ is absolutely convergent or conditionally convergent. Explain your reasoning.

Solution:
$\sum |a_k| = \sum_{k=1}^{\infty} \frac{k}{k^2+100}$ this series looks like $\sum_{k=1}^{\infty} \frac{k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k}$ which is a divergent p-series, $p = 1$.
Now $\lim_{k\to\infty} \frac{k}{k^2+100} \cdot \frac{k}{1} = 1 = L$ so $0 < L < \infty$, thus by the Limit Comparison Test the series $\sum |a_k| = \sum_{k=1}^{\infty} \frac{k}{k^2+100}$ diverges as well which implies that $\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^2+100}$ is conditionally convergent. [Remember is the series $\sum b_k$ converges, but the series $\sum |b_k|$ diverges the series is conditionally convergent.]

9. (12 points) Find the Taylor Series for $f(x) = \dfrac{1}{2x+2}$ at $a = 1$.

Solution:

k	$f^{(k)}(x)$
0	$(2x+2)^{-1}$
1	$-(2x+2)^{-2} \cdot 2$
2	$(1)(2)(2x+2)^{-3} \cdot 2 \cdot 2$
3	$-(1)(2)(3)(2x+2)^{-4} \cdot 2^2 \cdot 2$
4	$(1)(2)(3)(4)(2x+2)^{-5} \cdot 2^3 \cdot 2$

So $f^{(k)}(x) = (-1)^k k! 2^k (2x+2)^{-(k+1)}$

$f^{(k)}(1) = (-1)^k k! 2^k (2+2)^{-(k+1)} = \dfrac{(-1)^k k! 2^k}{4^{k+1}}$

And the Taylor Series is

$f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k k! 2^k}{4^{k+1}} \cdot \frac{1}{k!}(x-1)^k = \sum_{k=0}^{\infty} \frac{(-1)^k 2^k}{4^{k+1}}(x-1)^k$

10. (12 points) Determine the radius and interval of convergence for the series $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k\,3^k}$

Solution:

$$\lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)3^{k+1}} \cdot \frac{k3^k}{x^k} \right| = \frac{|x|}{3} \lim_{k\to\infty} \frac{k}{k+1} = \frac{|x|}{3} < 1$$

So the series converges if $-3 < x < 3$

Check the endpoints:

if $x = 3$ $\displaystyle\sum \frac{3^k}{k \cdot 3^k} = \sum \frac{1}{k}$ which is a divergent p-series since $p = 1$.

if $x = -3$ $\displaystyle\sum \frac{(-3)^k}{k \cdot 3^k} = \sum \frac{(-1)^k}{k}$ which is the alternating harmonic series that converges.

So the interval of convergence $-3 \le x < 3$ and the radius is 3.