1. Does the function $f(x)=|x|$ on $[-2,2]$ satisfy the conditions of the Mean Value Theorem? Why or why not?

Solution: No, because $f'(x)$ does not exist at $x=0$, therefore it is not differentiable on $(-2,2)$.

2. Determine the vertical and horizontal asymptotes of $f(x)=\dfrac{2x^2+6x}{x^2-9}$

Solution: $f(x)=\dfrac{2x(x+3)}{(x+3)(x-3)}$

$\lim\limits_{x\to 3^+}\dfrac{2x(x+3)}{(x-3)(x+3)}=\infty$ and $\lim\limits_{x\to 3^-}\dfrac{2x(x+3)}{(x-3)(x+3)}=-\infty$, thus we have a vertical asymptote at $x=3$

$\lim\limits_{x\to\pm\infty}\dfrac{2x^2+6x}{x^2-9}=2$, thus we have a horizontal asymptote at $y=2$.

3. Evaluate $\lim\limits_{x\to 0}\dfrac{e^{-5x}-1+5x}{x^2}$

Solution: $\lim\limits_{x\to 0}\dfrac{e^{-5x}-1+5x}{x^2}\overset{L}{=}\lim\limits_{x\to 0}\dfrac{-5e^{-5x}+5}{2x}\overset{L}{=}\lim\limits_{x\to 0}\dfrac{25e^{-5x}}{2}=\dfrac{25}{2}$

4. Evaluate $\lim\limits_{x\to\pi/2^-}\dfrac{3\sec x}{2+\tan x}$

Solution: $\lim\limits_{x\to\pi/2^-}\dfrac{3\sec x}{2+\tan x}\overset{L}{=}\lim\limits_{x\to\pi/2^-}\dfrac{3\sec x\tan x}{\sec^2 x}=\lim\limits_{x\to\pi/2^-}3\sin x=3$

5. Evaluate $\lim\limits_{x\to\pi/2^-}(1-\sin x)\tan x$

Solution: $\lim\limits_{x\to\pi/2^-}(1-\sin x)\tan x=\lim\limits_{x\to\pi/2^-}\dfrac{(1-\sin x)\sin x}{\cos x}\overset{L}{=}\lim\limits_{x\to\pi/2^-}\dfrac{\cos x-2\sin x\cos x}{-\sin x}=0$

6. Evaluate $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}-\dfrac{1}{x^2\sec x}\right)$

Solution:

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{x^2\sec x}\right)=\lim_{x\to 0}\left(\frac{\sec x-1}{x^2\sec x}\right)\overset{L}{=}\lim_{x\to 0}\left(\frac{\sec x\tan x}{2x\sec x+x^2\sec x\tan x}\right)$$

$$=\lim_{x\to 0}\left(\frac{\tan x}{2x+x^2\tan x}\right)\overset{L}{=}\lim_{x\to 0}\left(\frac{\sec^2 x}{2+2x\tan x+x^2\sec^2 x}\right)=\frac{1}{2}$$

7. Evaluate $\lim\limits_{x\to\infty}\left(1-\dfrac{4}{x}\right)^x$

Solution: $\lim\limits_{x\to\infty}\left(1-\dfrac{4}{x}\right)^x=\exp\left[\lim\limits_{x\to\infty}x\ln\left(1-\dfrac{4}{x}\right)\right]=e^{-4}$

8. Evaluate $\lim\limits_{x\to 0^+}(1+x)^{4/x}$

Solution: $\lim\limits_{x\to 0^+}(1+x)^{4/x}=\exp\left\{\lim\limits_{x\to 0^+}\dfrac{4}{x}\ln(1+x)\right\}\overset{L}{=}e^4$

9. Find the local and absolute extreme values of the function $f(x)=x-\sqrt{x}$ on [0, 4].

Solution: $f'(x)=1-\dfrac{1}{2\sqrt{x}}=\dfrac{2\sqrt{x}-1}{2\sqrt{x}}$ so $f'(x)=0$ when $x=\dfrac{1}{4}$, so $f(0)=0$, $f\left(\dfrac{1}{4}\right)=-\dfrac{1}{4}$ and $f(4)=2$.

Therefore absolute maximum occurs at $(4,2)$ and absolute minimum occurs at $\left(\dfrac{1}{4},-\dfrac{1}{4}\right)$.

10. Given $f'(x)=(x-1)(x+2)(x+4)$, determine the critical points of $f(x)$ and use the second derivative test to determine whether they correspond to local maxima, local minima, or the test is inconclusive.

Solution: $f'(x)=0$ at $x=1,-2,-4$. Now $f''(x)=(x-1)(x+2)+(x-1)(x+4)+(x+2)(x+4)$ and $f''(1)=15$, so at $x=1$ there is a local minimum, $f''(-2)=-6$, so at $x=-2$, there is a local maximum and $f''(-4)=10$, so at $x=-4$ there is a local minimum.

11. For each function, $f(x)=\left(x^2-1\right)^3$ and $f(x)=x\sqrt{3-x}$

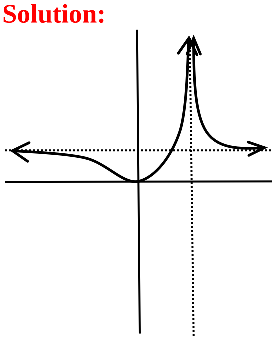
a. Find the critical points.
b. Find intervals of increase and decrease.
c. Find local maximum and minimum values.
d. Find intervals of concavity and inflection points.

For each function, $f(x)=\left(x^2-1\right)^3$ and $f(x)=x\sqrt{3-x}$

Solution: CP at $(0,-1)$, $(1,0)$, and $(-1,0)$, f is decreasing on $(-\infty,-1)\cup(-1,0)$ and increasing on $(0,1)\cup(1,\infty)$, local min at $(0,-1)$, and f is concave up on $(-\infty,-1)\bigcup\left(-\dfrac{1}{\sqrt{5}},\dfrac{1}{\sqrt{5}}\right)\bigcup(1,\infty)$ and concave down on $\left(-1,-\dfrac{1}{\sqrt{5}}\right)\bigcup\left(\dfrac{1}{\sqrt{5}},1\right)$

Solution: Note the domain is $(-\infty,3]$, CP $(2,2)$, $(3,0)$, f is increasing on $(-\infty,2)$ and decreasing on $(2,3)$, local max at $(2,2)$ and the function is concave down on $(-\infty,3)$

12. Sketch the graph of a function that satisfies all the conditions given below.
Solution:



13. A metal storage tank with volume V is to be constructed in the shape of a right circular cylinder surmounted by a hemisphere. What dimensions will require the least amount of metal? (The volume of a sphere is $\dfrac{4\pi r^3}{3}$.)

Solution: Volume: $V=\pi r^2 h+\dfrac{1}{2}\left(\dfrac{4}{3}\pi r^3\right)\Rightarrow h=\dfrac{V}{\pi r^2}-\dfrac{2r}{3}$, surface area is the sum of the area of the bottom, side, and top. $S(r)=\pi r^2+2\pi rh+\dfrac{1}{2}(4\pi r)=\dfrac{5}{3}\pi r^2-\dfrac{2V}{r}$, for $r>0$ and $S'(r)=\dfrac{10\pi r^3-6V}{3r^2}=0$ when $r=\sqrt[3]{\dfrac{3V}{5\pi}}$, and $S''(r)=\dfrac{10\pi}{3}-\dfrac{4V}{r^3}<0$ when $r=\sqrt[3]{\dfrac{3V}{5\pi}}$ thus the dimensions that minimize surface area are $r=\sqrt[3]{\dfrac{3V}{5\pi}}=h$

14. An inverted conical tank with height 10 feet and radius 4 feet is full of water. Water drains from the tank at the rate of 5 ft^3 /min, how fast is the water level dropping when the height is 6 feet.

Solution: $V=\frac{1}{3}\pi r^2 h$ so $\frac{dV}{dt}=\frac{2}{3}\pi r\frac{dr}{dt}h+\frac{1}{3}\pi r^2\frac{dh}{dt}$, also $\frac{r}{h}=\frac{4}{10}\Rightarrow r=\frac{2}{5}h$ and $\frac{dr}{dt}=\frac{2}{5}\frac{dh}{dt}$

So $\frac{dV}{dt}=\frac{2}{3}\pi r\frac{dr}{dt}h+\frac{1}{3}\pi r^2\frac{dh}{dt}=\frac{2}{3}\pi\left(\frac{2}{5}h\right)\left(\frac{2}{5}\frac{dh}{dt}\right)h+\frac{1}{3}\pi\left(\frac{2}{5}h\right)^2\frac{dh}{dt}$

$$-5=\frac{2}{3}\pi\left(\frac{2}{5}(6)\right)\left(\frac{2}{5}\frac{dh}{dt}\right)6+\frac{1}{3}\pi\left(\frac{2}{5}6\right)^2\frac{dh}{dt}=\frac{64\pi}{25}\frac{dh}{dt},\quad \frac{dh}{dt}=-5\left(\frac{25}{64\pi}\right)$$

15. A closed box with square base is to be built to house an ant colony. The bottom of the box and all four sides are to be made of material costing $\$1/\text{ft}^2$ and the top is to be constructed of glass costing $\$5/\text{ft}^2$. What are the dimensions of the box of greatest volume that can be constructed for \$72? Verify your answer yields a maximum.

Solution: Cost $C=x^2+4xy+5x^2=6x^2+4xy=72$, so $y=\frac{72-6x^2}{4x}$ for $x>0$. Now $V=x^2y=x^2\left(\frac{72-6x^2}{4x}\right)=18x-\frac{9}{2}x^3$ and $V'=18-\frac{9}{2}x^2=0$ when $x=2$. To verify that $x=2$ maximizes volume, use the second derivative test, $V''=-9x$ so $V''(2)=-18$ thus volume is maximized when $x=2, y=6$.

16. A viewer standing 30 feet from a platform watches a balloon rise from that platform (the platform is the same height as the viewer's eyes) at a constant rate of 3 ft/s. How fast is the angle between the viewer and the balloon changing at the instant $\theta=\frac{\pi}{4}$.

Solution: $\tan\theta=\frac{y}{30}\Rightarrow\sec^2\theta\frac{d\theta}{dt}=\frac{1}{30}\frac{dy}{dt}$, so $\frac{d\theta}{dt}=\frac{1}{30}(3)\frac{1}{\sec^2(2)}=\frac{1}{20}$

For additional problems, check out the review problems for Chapter 3. Note the questions above are simply a sample of questions possible for the exam; it is possible that other types of questions may appear on your exam.