Achievement and Avoidance Games on Semigraphs

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Abstract

A semiline $h = (u, v)$ is "one-half " of an edge that is incident to u but not to v. One can imagine that h goes "halfway" from u to v. The edge uv consists of the two semilines (u, v) and (v, u). Starting with four vertices, two players, A and B, take turns adding semilines. In the triangle achievement (avoidance) game, the first player to complete a triangle wins (loses). We will show that player B has a winning strategy in both games. We will also consider quadrilateral achievement and avoidance games on four and five vertices.

Introduction

Definition 1: A semigraph $\mathbf{S}$ has vertex set $V(\mathbf{S})$ and edge set $H(\mathbf{S})$. Each semiline $h \in H(\mathbf{S})$ has the form (u, v), where h is incident to u but not v. In this sense one can imagine that h goes "half way" to vertex v. The two semilines $h_1 = (u, v)$ and $h_2 = (v, u)$ combine to make an edge uv joining vertices u and v.

Example 1: A semigraph of order four with seven semilines.

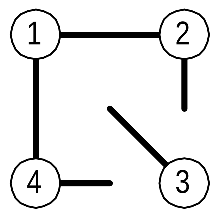

Figure 1

Consider the following game :

(1) Starting with four vertices, two players, A and B, take turns adding semilines to the graph.

(2) The first player to complete a triangle is the winner.

This game is called the triangle achievement game on four vertices. In the triangle avoidance game the first player to complete a triangle is the loser. In the quadrilateral achievement (avoidance) game, the first player to complete a quadrilateral wins (loses). These and similar games can also be played with an arbitrary number of vertices.

Theorem 1: Player B can win the triangle and quadrilateral achievement games on any number of vertices by simply completing each semiline started by player A. ■

Theorem 2: In the triangle avoidance game on four vertices player B has a winning strategy.

Proof : Let the vertices be denoted

$$\{1, 2, 3, 4\}$$

We now describe the winning strategy for player B : Let

$$\{i, j, r, s\} = \{1, 2, 3, 4\} .$$

If player A adds semiline (i, j) then player B adds semiline (r, s) or (s, r). To prove that this strategy works, notice that there are four possible triangles

$$T_1 = 234, T_2 = 134, T_3 = 124 \text{ and } T_4 = 123.$$

The important facts are

(a) Triangle T_k is missing edge k.

(b) Edge rs is common to T_i and T_j.

Thus, after the first round each of the four triangles is incident with exactly one semiline. After round k each triangle is incident with k semilines. If player B uses this strategy then the game will continue to round five with each triangle incident with five semilines. In round 6 player A will add a semiline and complete two triangles, according to Property (b). Therefore, player B will win. ■

Theorem 3: Player B has a winning strategy in the quadrilateral avoidance game on four vertices.

Proof : This game must have at least four rounds and terminates in round four only if player B completes a quadrilateral. Since this is easily avoided, we can assume that the game has at least five rounds. In round five player A will add one semiline to create a graph with 9 semilines. Now, every graph of order 4 with 9 semilines corresponds to a digraph of order 4 with 9 arcs. Using Appendix 2 in [2], we can enumerate the 13 semigraphs of order 4 with 9 semilines.

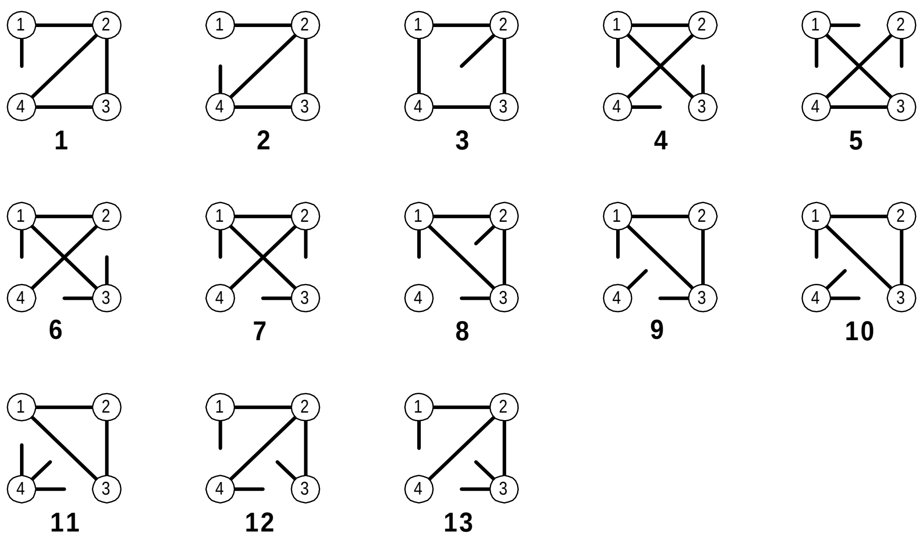

Figure 2 : The 13 semigraphs of order 4 with nine semilines, S_1 to S_{13}.

If player A creates a graph isomorphic to S_3, then the game is over and player B wins. Thus, we can assume that player A created a graph isomorphic to one of the other twelve graphs in Figure 2. There are exactly five semigraphs of order 4 with ten semilines

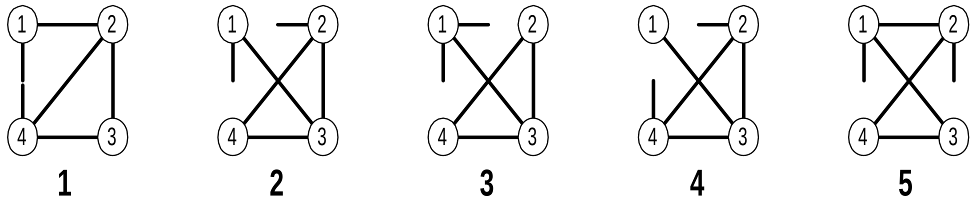

Figure 3 : The five semigraphs of order 4 with ten semilines, S_1 to S_5.

It is clear that player B wants to create a semigraph isomorphic to S_2, S_3 or S_4 in Figure 3. It can be easily verified that each semigraph in Figure 2 except S_3 is a subgraph of either S_2, S_3 or S_4 in Figure 3. Furthermore, any semiline added to a graph of Figure 3 will create a semigraph with at least one quadrilateral. Therefore, player B has as winning strategy : **Add semilines that do not create a quadrilateral and win the game in round five or six.** ■

The Triangle Avoidance Game on five vertices

Definition 2: We will denote the vertex set of S by

$$V(S) = \{1, 2, 3, 4, 5\}.$$

S^1 denotes the semigraph of the game after player A adds a semiline in round 1, S^2 is the semigraph of the game after player B adds a semiline in round 1 and so on.

Suppose that S^{13} (the graph created by player A in round 7) contains a triangle, T, and the game terminates. A triangle has 3 edges and there are 7 missing edges . Since S^{13} is formed by adding exactly seven semilines to T there is at least one pair of vertices jk that are not incident to any semiline of S^{12}. Thus, player A could have found a safe move to continue the game. Let's consider the moves of player B in the first six rounds.

Lemma 1: Player B has a strategy so that one the following is true

(a) S^{10} is acyclic

(b) S^{10} has a 4-cycle but no 5-cycle.

Proof : Keep in mind that if a 5-cycle appears in S^k then player B will eventually lose. Moreover, if S^k has a 4-cycle and a 5-cycle then it must contain a 3-cycle. Thus, player B has a chance to win if he can create a 4-cycle in S^k before the appearance of a 5-cycle or keep S^k acyclic. Let's consider the following strategy for player B :

(1) In the first three rounds complete every semiline started by player A except in the case when a triangle is formed.

(2) There are four possible graphs created by the above strategy.

Assume that S^6 is isomorphic to the following graph

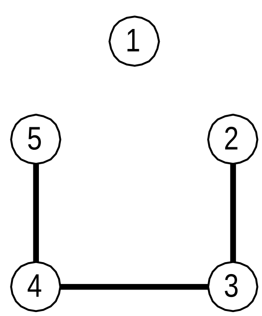

Figure 4 :

(**CLAIM**) Player B can complete the 4-cycle 2345 regardless of what player A does and before player A can create a 5-cycle : Player B has to complete the edge 25 but player A must add at least two edges to complete a 5-cycle. Player A may attempt to add semilines and prevent player B from completing the 4-cycle 2345 but this can only happen if the addition of edge 25 creates a triangle among the vertices 2,3,4 and 5. Notice that the resulting semigraph must have two triangles! Thus, at least one of triangle was completed during player A's turn! Thus, player B can complete the 4-cycle 2345 and prevent player A from completing a 5-cycle in subsequent rounds.

Case 2: Assume that S^6 is isomorphic to the following graph

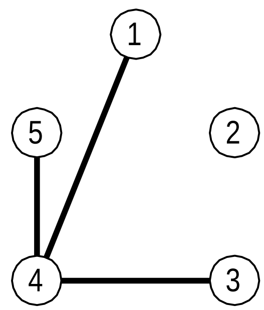

Figure 5 :

If a 5-cycle is created in S^k, $k > 6$, then a 3-cycle is also created because one of the whole edges 45,41 or 43 will be a chord of the 5-cycle.

Case 3: Assume that S^6 is isomorphic to the following graph

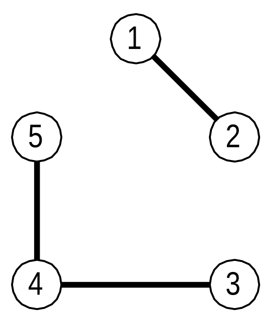

Figure 6 :

Player B should make sure that either 41 or 42 is added to the graph.

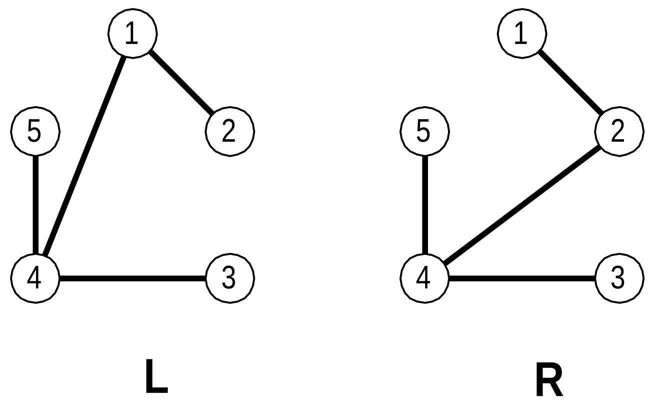

Figure 7 :

This prevents player A from completing a 5-cycle in subsequent rounds.

Case 4: Player B should create the following graph in round 3 instead of a triangle.

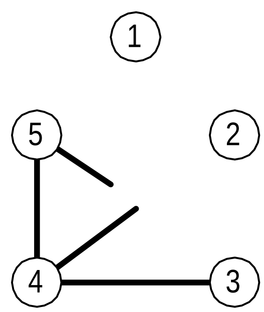

Figure 8 :

Now player B can insure that edge 42 is in the graph.

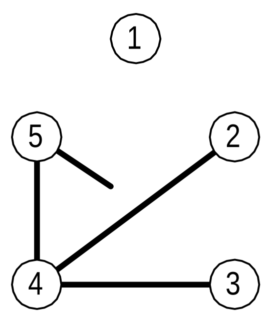

Figure 9 :

This prevents player A from completing a 5-cycle in subsequent rounds. ■

Definition 3: Let **S** be a semigraph. The spanning subgraph of **S** consisting of the whole edges is called the underlying whole edge subgraph and is denoted $\mathbf{S}_W$.

Example 2: A semigraph **S** and its underlying whole edge subgraph $\mathbf{S}_W$.

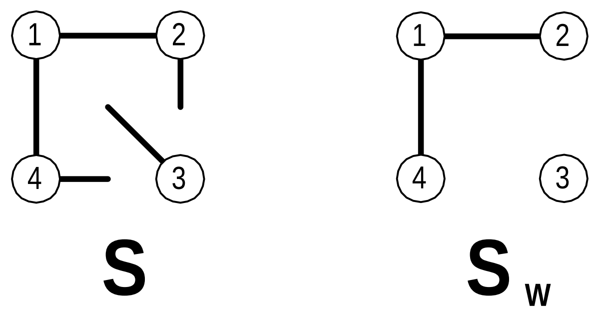

Figure 10 :

Theorem 4: Player B has a winning strategy in the triangle avoidance game on five vertices.

Proof : Assume that player A has winning strategy and that player B follows the strategy of Lemma 1. Since there are ten pairs of distinct vertices, the game should not end before round 6. By Turan's Theorem [2], every graph with five vertices and seven edges has a triangle. Therefore we conclude that S^{17} has a triangle and the game terminates on or before its completion. Since player B is forced to create a triangle S_W^{11}, S_W^{13} or S_W^{15} is one of the following maximal triangle-free graphs.

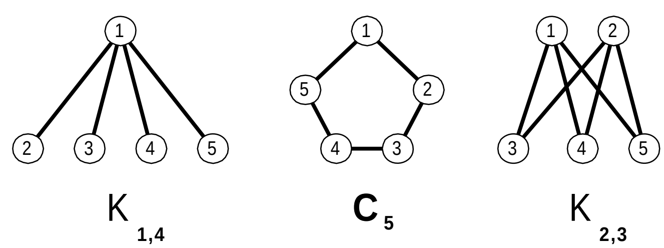

Figure 11 :

See [Theorem 3, 1]. By using the strategy of Lemma 1, player B has eliminated C_5 from consideration.

Case 1: (S^{12} contains a triangle) S_W^{11} is isomorphic to $K_{1,4}$ or $K_{2,3}$.

Case 1a: (S_W^{11} is isomorphic to $K_{1,4}$) S_W^{11} has six missing edges and exactly three semilines must be added to create S^{11}. Thus, there are three different semilines that player B can add to S^{11} that will not create a triangle.

Case 1b: (S_W^{11} is isomorphic to $K_{2,3}$) This case is impossible because $K_{2,3}$ has 12 semilines.

Case 2: (S^{14} contains a triangle) S_W^{13} is isomorphic to $K_{1,4}$ or $K_{2,3}$.

Case 2a: (S_W^{13} is isomorphic to $K_{1,4}$) S_W^{13} has six missing edges and exactly five semilines must be added to create S^{13}. Thus, there is one semiline that player B can add to S^{13} that will not create a triangle.

Case 2b: (S_W^{13} is isomorphic to $K_{2,3}$) S_W^{13} has four missing edges and exactly one semiline must be added to create S^{13}. Thus, there are three different semilines that player B can add to S^{13} that will not create a triangle.

Case 3: (S^{16} contains a triangle) S_W^{15} is isomorphic to $K_{1,4}$ or $K_{2,3}$.

Case 3a: (S_W^{15} is isomorphic to $K_{1,4}$) S_W^{15} has six missing edges but seven semilines must be added to create S^{15}. Thus one whole edge is created and S^{15} has a triangle. Therefore the game is already over !

Case 3b: (S_W^{15} is isomorphic to $K_{2,3}$) S_W^{15} has four missing edges and exactly three semilines must be added to create S^{15}. Thus, there is one semiline that player B can add to S^{15} that will not create a triangle. We have shown that in all possible cases player A does not have a strategy that forces player B to create a triangle. Therefore, if player B follows the strategy of Lemma 1 then player A has no winning strategy. This completes the proof of the theorem.

Player B can win with the following strategy :

(a) Follow the strategy of Lemma 1 for the first 5 rounds.

(b) In rounds 6 through 8 there will be at least one safe move.

(c) Since S^{17} has a triangle, the game will be over on or before its creation.

Summary

In the achievement (avoidance) games two players start with an empty graph and take turns adding semilines. The first player to complete cycle C_k is the winner (loser).

(1) Player B can win the C_k achievement game by completing each semiline started by player A.

(2) Player B can win the triangle avoidance game on four vertices by using the following strategy : If player A adds semiline (i, j) then player B adds semiline (r, s) or (s, r), where

$$\{i, j, r, s\} = \{1, 2, 3, 4\} .$$

(3) Player B can win the quadrilateral avoidance game on four vertices by using the following strategy : Add semilines that do not create a quadrilateral and win the game in round five or six.

(4) Player B can win the triangle avoidance game on five vertices by using the following strategy :

(a) Follow the strategy of Lemma 1 for the first 5 rounds.

(b) In rounds 6 through 8 there will be at least one safe move.

(c) Since S^{17} has a triangle, the game will be over on or before its completion.

Computational Results

The avoidance game becomes more complicated as the order of the graph increases but we have used a computer to verify the following :

The Triangle Avoidance game on 6 vertices and the Quadrilateral Avoidance game on 5 vertices have winning strategies for player B.

References

[1] C. Barefoot, K. Casey, D. Fisher, K. Fraughnaugh, F. Harary, **Size in maximal triangle-free graphs and minimal graphs of diameter 2**, *Discrete Math.* 138 (1995) 93-99.

[2] F. Harary, *Graph Theory*, Addison Wesley, Reading MA (1969).